Chapter 1
MEASUREMENTS AND VECTORS

1.1 UNITS AND STANDARDS

Any physical quantity must have, besides its numerical value, a standard unit.

In mechanics all quantities are derived from the three-fundamental quantities: **mass**, **length**, and **time**.

Several systems of units are used in physics:

	Length	Mass	Time
International System (SI) (mks)	meter (m)	kilogram (kg)	second (s)
Gaussian System (cgs)	centimeter (cm)	gram (g)	second (s)
British Engineering System	foot (ft)	slug	second (s)

If different unit systems are used in a physical equation one system should be chosen and the quantities with the other unit systems must be converted to the chosen system.

1.2 DIMENSIONAL ANALYSIS

Dimension in physics gives the physical nature of the quantity, whether it is a length (L), mass (M), or time (T).

All other quantities in mechanics can be expressed in terms of these fundamental quantities.

Quantity	Dimension	Un it (mks, cgs, british)
Area	L^2	m^2, cm^2, ft^2
Velocity	L/T	m/s, cm/s, ft/s
Force	ML/T^2	Newton, dyne, lb
Enenrgy	ML^2/T^2	Joule, erg, lb.ft

Example 1.1 Show that the equation below is dimensionally correct.

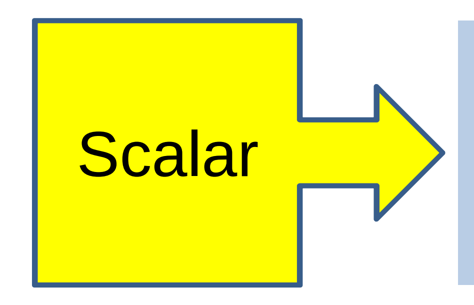
$$x = v_o t + \frac{1}{2} a t^2$$

Solution Since

$$[x] = L \qquad [v] = L/T \qquad [a] = L/T^2$$

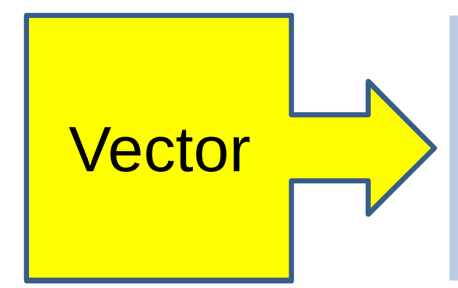
$$L = \frac{L}{T} T + \frac{L}{T^2} T^2 = L$$

1.3 VECTORS and SCALARS



is the physical quantity that has magnitude only.

e.g., time, volume, mass, density, energy, distance, temperature.



is the physical quantity that has both magnitude and direction.

e.g., displacement, velocity, acceleration, force, area.

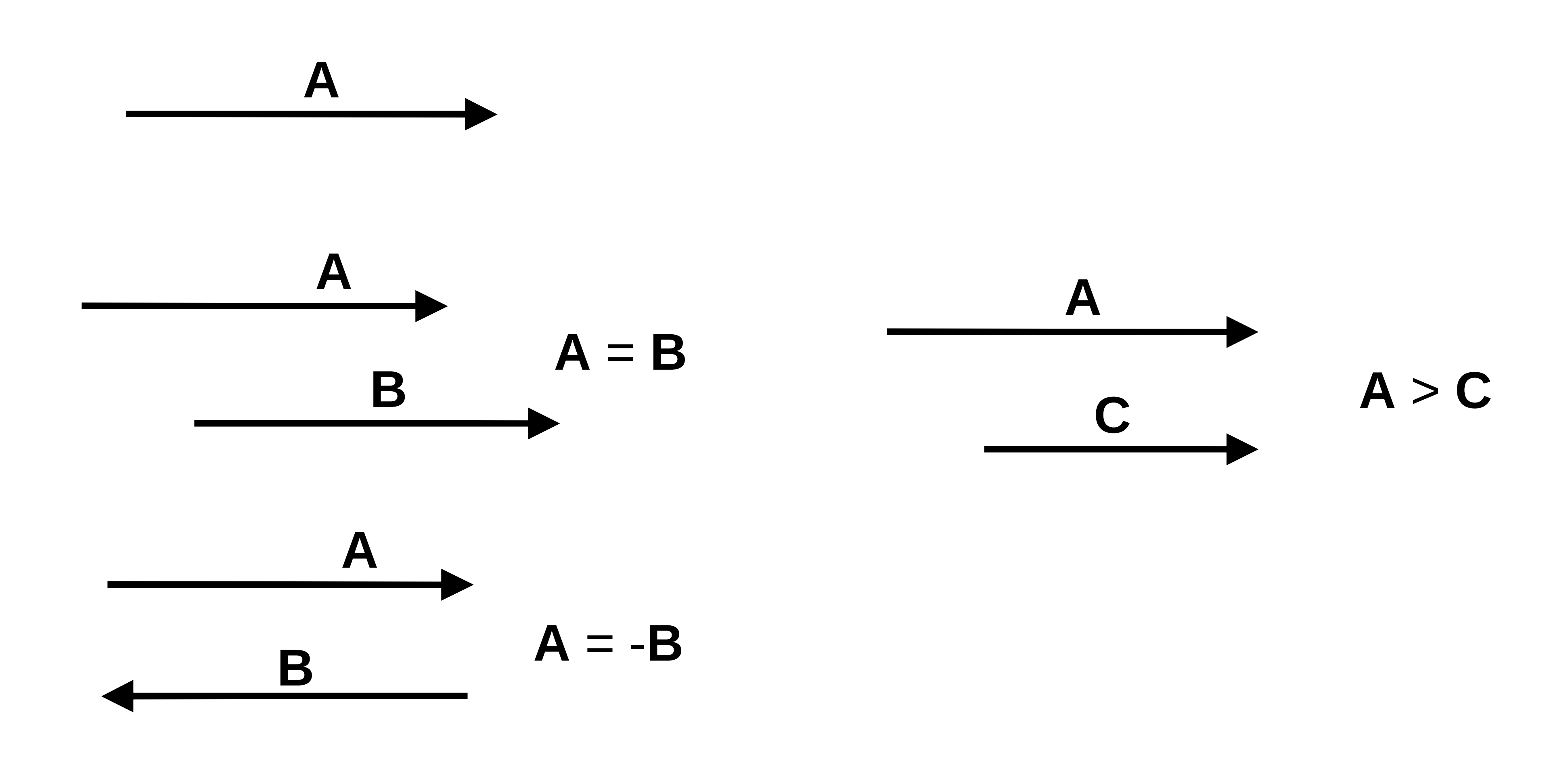
The vector quantity will be distinguished from the scalar quantity by typing it in boldface, like ***A***. In write handing the vector quantity is written with an arrow over the symbol, such as, $\vec{A}$ The magnitude of the vector **A** will be denoted by $|\vec{A}|$, or simply the italic type A.

Geometrically, The vector quantity is represented by a straight line and an arrow at one end of the line.

Tail of the vector → Head of the vector

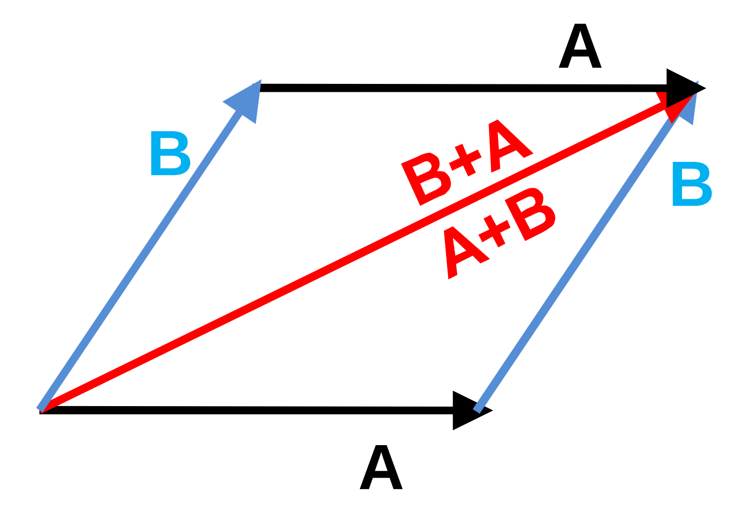
Equality of two vectors

Any two vectors are said to be equal if they have the same magnitude and point in the same direction, regardless of their location.

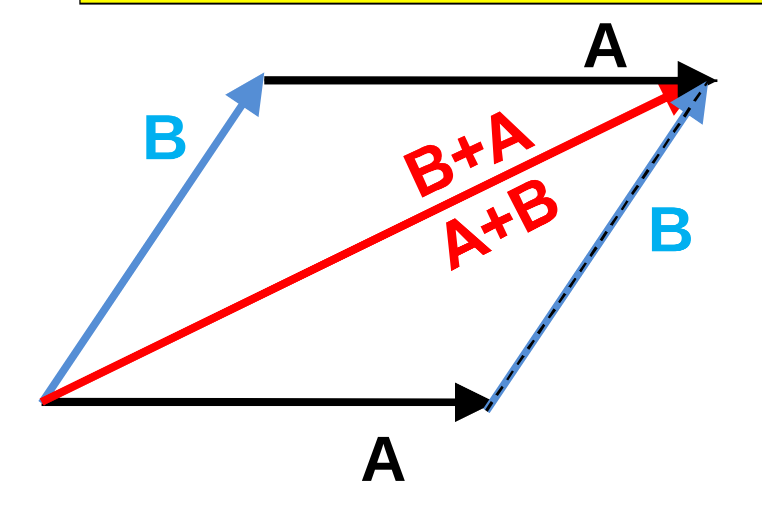


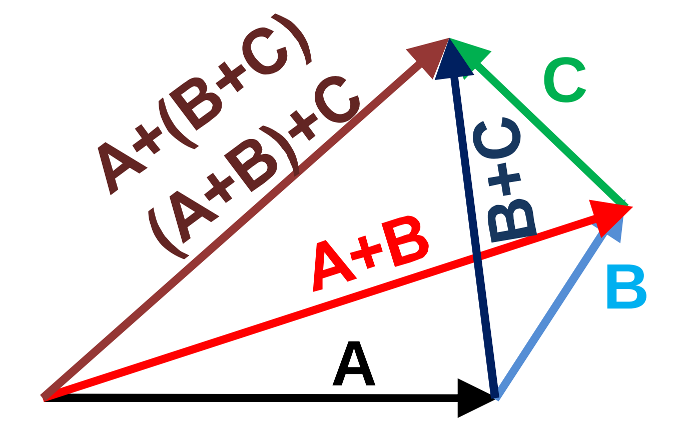
Addition of vectors (graphical method)

triangle method.



parallelogram method.



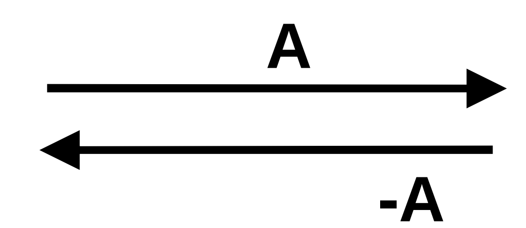


A+**B** = **B**+**A** (commutative)

(**A**+**B**)+**C** = **A**+(**B**+**C**) (associative)

Negative of a vectors

The vector **-A** is a vector with the same magnitude as the vector **A** but points in opposite direction.



Subtraction of vectors

Subtracting vector **B** from vector **A** is the same as adding **-B** to **A**, i.e., **A**-**B**=**A**+(-**B**)

UNIT VECTOR

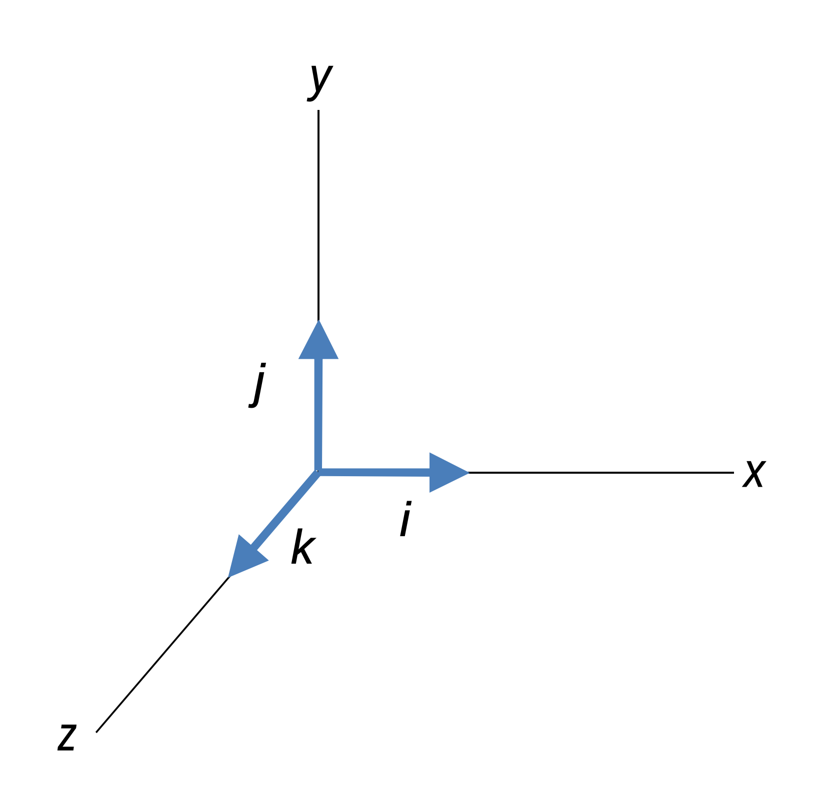
A unit vector is a vector, with specific direction, having a magnitude of unity without any units or dimensions.

In Cartesian coordinates three unit vectors are adopted to specify the positive directions of the three-axes.

i is directed along the positive *x*-axis

j is directed along the positive *y*-axis

k is directed along the positive *z*-axis



This means that if a vector **A** is directed a long the positive *x*-axis with a magnitude of *A*, this vector can be written as **A**=*A***i**.

The vector **B**=*B***j** means that it has a magnitude *B* and is directed a long the positive *y*-axis.

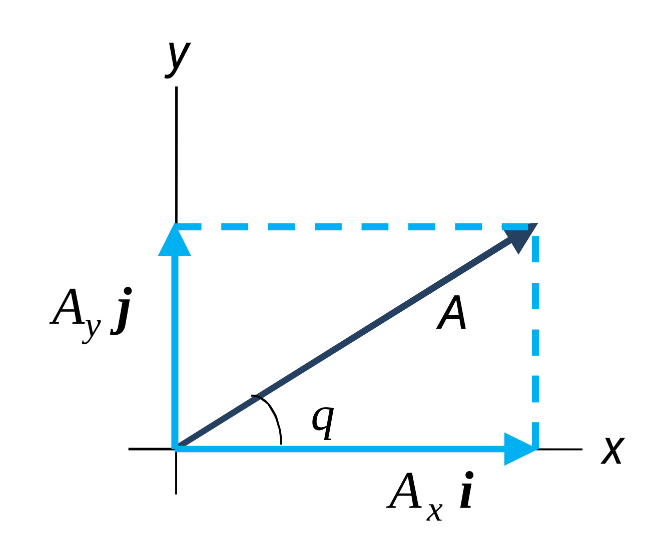
And the vector **C**=*C***k** means that it has a magnitude *C* and is directed a long the positive *z*-axis.

The minus sign in front of any vector indicates the opposite direction of that vector, i.e. **-i** refers to the negative *x*-axis, and so for the other two unit vectors.

COMPONENTS OF VECTOR

consider the vector **A** in the *x*-*y* plane.

It is clear that the vector **A** can be represented by the sum of two vectors, one parallel to the *x*-axis ($A_x\,\boldsymbol{i}$), and the other parallel to the *y*-axis ($A_y\,\boldsymbol{j}$), i.e.,



$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j}$$

A_x and A_y are called, respectively the *x*- and the *y*-components of the vector $\boldsymbol{A}$.

It is clear from the figure that $A_x = A\cos q$ and $A_y = A\sin q$

The magnitude of the vector is given by $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$

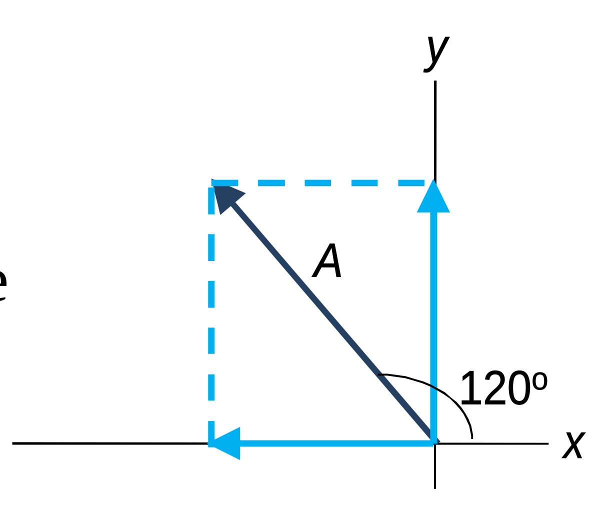
In general any vector $\boldsymbol{A}$ can be resolved into three components as.

$$\boldsymbol{A} = A_{\mathrm{x}}\mathbf{i} + A_{\mathrm{y}}\mathbf{j} + A_{\mathrm{z}}\mathbf{k}$$

Where A_z is called the z-components of the vector $\boldsymbol{A}$.

Note that if the vector $\boldsymbol{A}$ is zero then all its components are zero.

Example 1.2 A vector **A** lying in the *x*-*y* plane has a magnitude *A*=50.0 units and is directed at an angle of 120º to the positive *x*-axis, as shown in the figure. What are the rectangular components of this vector?



Solution From the figure it is clear that the vector A has 2-components: one along the -ve *x*-axis and the other along the +ve *y*-axis. Now

$$A_x = A\cos q = 50\cos 120^o = -25.0\,\text{units}$$

$$A_y = A\sin q = 50\sin 120^o = 43.3\,\text{units}$$

ADDING VECTORS

To add two vectors analytically proceed as follow:

(i) Resolve each vector into its components according to suitable coordinate axes.

(i) Add, algebraically, the *x*-components of the individual vectors to obtain the *x*-component of the resultant vector. Do the same thing for the other components, i.e., if

$$\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

and $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$

Then the resultant vector $\boldsymbol{R}$ is

$$\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Example 1.3 If $\boldsymbol{A} = 4\mathbf{i} + 3\mathbf{j}$ and $\boldsymbol{B} = -3\mathbf{i} + 7\mathbf{j}$

find the resultant vector **R= A+B.**

Solution From the given information we have

$$A_x = 4, \qquad B_x = -3, \qquad A_y = 3 \text{ and } B_y = 7$$

$$\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} \quad \Rightarrow$$

$$\boldsymbol{R} = (4-3)\mathbf{i} + (3+7)\mathbf{j} = \mathbf{i} + 10\mathbf{j}$$

Example 1.4 A particles undergoes three consecutive displacenents given by

$$\boldsymbol{d}_1 = (\mathbf{i} + 3\mathbf{j}\text{-}\mathbf{k})cm, \quad \boldsymbol{d}_1 = (2\mathbf{i} - \mathbf{j}\text{-}3\mathbf{k})cm, \quad \text{and} \quad \boldsymbol{d}_1 = (\text{-}\,\mathbf{i} + \mathbf{j})cm,$$

find the resultant displacement of the particle.

Solution

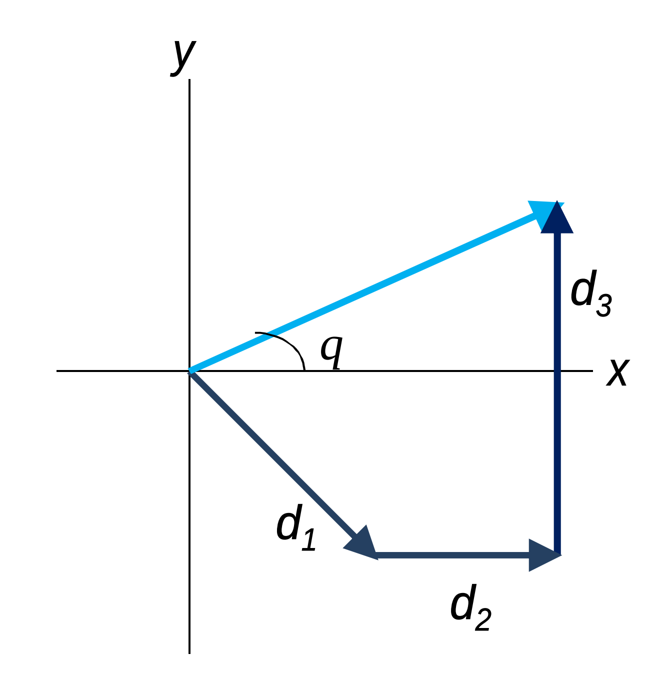
$$\boldsymbol{R} = \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3$$

$$= (d_{1x} + d_{2x} + d_{3x})\mathbf{i} + (d_{1y} + d_{2y} + d_{3y})\mathbf{j} + (d_{1z} + d_{2z} + d_{3z})\mathbf{k} \Rightarrow$$

$$\boldsymbol{R} = (1 + 2 - 1)\mathbf{i} + (3 - 1 + 1)\mathbf{j} + (-1 - 3 + 0)\mathbf{k}$$

$$\boldsymbol{R} = (2\mathbf{i} + 3\mathbf{j} + -4\mathbf{k})cm$$

Example 1.5 A particle undergoes the following consecutive displacements: 4.3 m southeast, 2.4 m east, and 5.2 m north. Find the magnitude and the direction of the resultant vector.



Solution

The three displacements can be written as.

$$\boldsymbol{d}_1 = (4.3\cos 45)\mathbf{i} - (4.3\sin 45)\mathbf{j} = (3.04\mathbf{i} - 3.04\mathbf{j})\text{m}$$

$$\boldsymbol{d}_2 = 2.4\mathbf{i}\text{ m} \qquad \text{and } \boldsymbol{d}_3 = 5.2\mathbf{j}\text{ m}$$

$$\text{Now } \boldsymbol{R} = \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 \quad = (3.04 + 2.40 + 0)\mathbf{i} + (-3.04 + 0 + 5.20)\mathbf{j}$$

$$\Rightarrow \quad \boldsymbol{R} = (5.44\mathbf{i} + 2.16\mathbf{j})\text{m}$$

To find the magnitude of the resultant vector we have.

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(5.44)^2 + (2.16)^2} = 5.85\text{m}$$

To find the direction of a vector it is enough to determine the angle the vector makes with a specific axis. The angle q makes with the *x*-axis is

$$q = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{2.16}{5.44}\right) = 21.66^{o}$$

Example 1.6 If $\boldsymbol{B} = 4\mathbf{i} + 3\mathbf{j}$

find the vector $\boldsymbol{A}$ such that $\boldsymbol{A} + \boldsymbol{B} = 5\mathbf{i}$

Solution From the given data we conclude

$$A_x + B_x = 5 \qquad \text{and} \qquad A_y + B_y = 0 \qquad \Rightarrow$$

$$A_x = 5 - B_x = 5 - 4 = 1 \qquad \text{and } A_y = -B_y = 1$$

$$\Rightarrow \quad \boldsymbol{A} = \mathbf{i} + \mathbf{j}$$

PRODUCTS OF VECTOR

1- Multiplying a vector by a scalar:

If l is a scalar and $\boldsymbol{A}$ is a vector then

$$l\boldsymbol{A} = lA_x\mathbf{i} + lA_y\mathbf{j} + lA_z\mathbf{k}$$

Note that the direction of $l\,\boldsymbol{A}$ is the same as the direction of $\boldsymbol{A}$ if l is +ve a and opposite to A if l is –ve.

2- Scalar (dot) product:

The scalar product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A}\cdot\boldsymbol{B}$, is a **scalar** quantity defined by,

$$\boldsymbol{A}\cdot\boldsymbol{B} = ABcosq$$

with q is the smallest angle between the two vectors.

Since **i**, **j**, and **k** are unit vectors perpendicular to each other, and using the last equation we can write

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

Now if $\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} \Rightarrow$

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\cdot\left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)\\ &= A_xB_x\mathbf{i}\cdot\mathbf{i} + A_xB_y\mathbf{i}\cdot\mathbf{j} + A_xB_z\mathbf{i}\cdot\mathbf{k}\\ &+ A_yB_x\mathbf{j}\cdot\mathbf{i} + A_yB_y\mathbf{j}\cdot\mathbf{j} + A_yB_z\mathbf{j}\cdot\mathbf{k}\\ &+ A_zB_x\mathbf{k}\cdot\mathbf{i} + A_zB_y\mathbf{k}\cdot\mathbf{j} + A_zB_z\mathbf{k}\cdot\mathbf{k}\end{aligned}$$

1 1 1

$$\Rightarrow \boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

Example 1.7 If $\boldsymbol{A} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\boldsymbol{B} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

find the angle between the two vectors ***A*** and ***B*.**

Solution Since $\boldsymbol{A}\cdot\boldsymbol{B} = ABcosq \Rightarrow \cos\theta = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB}$

But $\boldsymbol{A}\cdot\boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z = 2 + -6 - 6 = -10$

$A = \sqrt{1+4+9} = 3.74$ and $B = \sqrt{4+9+4} = 4.12$

$\Rightarrow \cos\theta = \frac{-10}{(3.74)(4.12)} = -0.69 \Rightarrow q = 134^o$

Example 1.8 Consider the two vectors given in the previous example. Find $2\boldsymbol{A}\cdot\boldsymbol{B}$,

Solution From the previous example we have

$$\boldsymbol{A}=\mathbf{i}-2\mathbf{j}+3\mathbf{k} \qquad \text{and } \boldsymbol{B}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k} \qquad \Rightarrow$$

$$2\boldsymbol{A}=2\mathbf{i}-4\mathbf{j}+6\mathbf{k}$$

Then $2\boldsymbol{A}\cdot\boldsymbol{B}=4+-12-12=-20$

Note that if one multiply the product $\boldsymbol{A}\cdot\boldsymbol{B}$ by 2 or multiply $\boldsymbol{A}\cdot\boldsymbol{2B}$ we get the same result, i.e.,

$$2(\boldsymbol{A}\cdot)=(2\boldsymbol{A})\cdot\boldsymbol{B}=\boldsymbol{B}\cdot(2\boldsymbol{A})$$

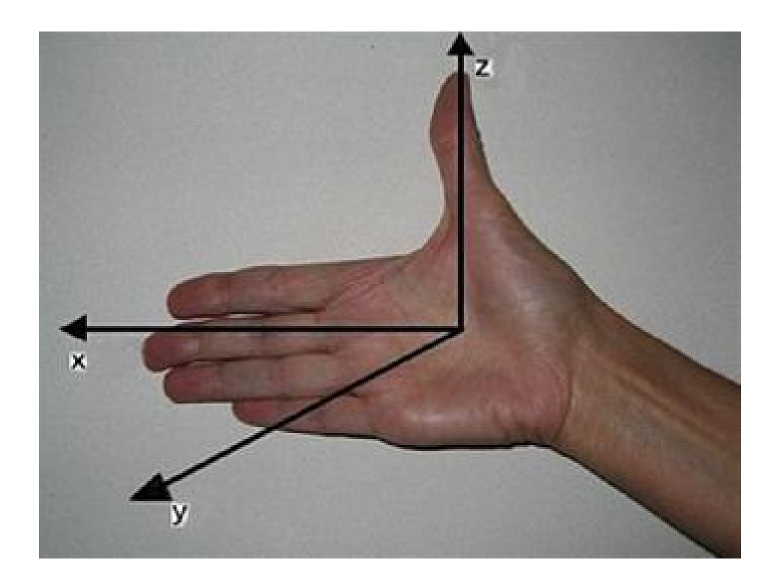
3- Vector (cross) product:

The vector product of two vectors ***A*** and ***B***, written as ***A*** ´ ***B***, is a third vector **C** with a magnitude given by

$$C = |\boldsymbol{A} \times \boldsymbol{B}| = ABsinq$$

The vector ***C*** is perpendicular to the plane of ***A*** and ***B***. Since there are two directions perpendicular to a given plane, the **right hand rule** is used to decide to which direction the vector ***C***= ***A*** ´ ***B*** is directed.

The rule states that the four fingers of the right hand are pointed along ***A*** and then curled toward ***B*** through the smaller angle between ***A*** and ***B***. The thumb then gives the direction of ***C***.



Using this rule it is clear that.

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

By the same rule one can verify that

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

Now if $\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$ $\Rightarrow$

$$\boldsymbol{A}\times\boldsymbol{B} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\times\left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)$$

$$= A_x B_x \mathbf{i}\times\mathbf{i} + A_x B_y \mathbf{i}\times\mathbf{j} + A_x B_z \mathbf{i}\times\mathbf{k}$$

$$+ A_y B_x \mathbf{j}\times\mathbf{i} + A_y B_y \mathbf{j}\times\mathbf{j} + A_y B_z \mathbf{j}\times\mathbf{k}$$

$$+ A_z B_x \mathbf{k}\times\mathbf{i} + A_z B_y \mathbf{k}\times\mathbf{j} + A_z B_z \mathbf{k}\times\mathbf{k}$$

$$= (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

Or equivalently we have

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=(A_y\mathrm{B}_z - \mathrm{A}_z B_y)\mathbf{i}+(A_z\mathrm{B}_x - \mathrm{A}_x B_z)\mathbf{j}+(A_x\mathrm{B}_y - \mathrm{A}_y B_x)\mathbf{k}$$

Example 1.9 Consider the two vectors given in the previous example. Find ***A ´B*** ,

Solution From the previous example we have

$$\boldsymbol{A}=\mathbf{i}-2\mathbf{j}+3\mathbf{k} \qquad \text{and } \boldsymbol{B}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k} \qquad \Rightarrow$$

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 3\\ 2 & 3 & -2\end{vmatrix}=(4-9)\mathbf{i}+(6+2)\mathbf{j}+(3+4)\mathbf{k}$$

$$=-5\mathbf{i}+8\mathbf{j}+7\mathbf{k} \quad \Rightarrow$$

Chapter 2
Linear Motion

2.1 Average Velocity

The average velocity of a particle is defined as

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}}$$

The average velocity depends only on the initial and the final positions of the particle.

This means that if a particle starts from a point and return back to the same point, its displacement, and so its average velocity is zero.

Remark: There is a difference between distance and displacement.

Distance, a scalar quantity, is the actual long of the path traveled by a particle, but **displacement**, a vector quantity, is the shortest distance between the initial and the final positions of the particle.

Speed is the magnitude of the velocity. This means that the speed can never be negative.

The average speed differs from average velocity in that it covers the total distance rather than the total displacement, that is

$$\text{average speed} = \frac{\text{total distance}}{\Delta t}$$

The SI unit of velocity is m/s.

2.1 Instantaneous Velocity

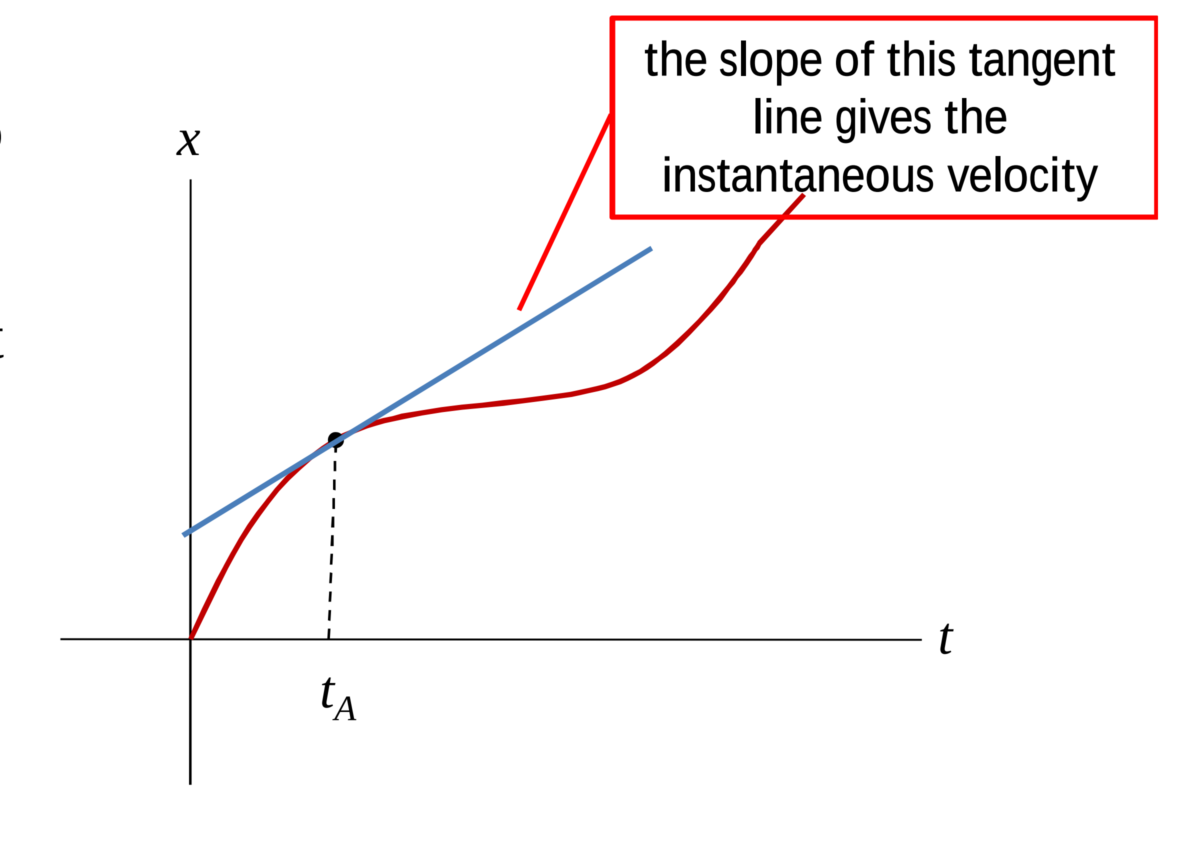
The instantaneous velocity of a particle is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \quad \Longrightarrow \quad v = \frac{dx}{dt}$$

In the position-time graph v at some instant is the slope of the tangent at that instant.

Let the position of a particle varies with time according to the graph shown

The instantaneous velocity at specific time t_A is the slope of the tangent at that point.



Example 2.1 The position of a particle varies with time according to $x = t^2 + 3t$ with x in m and t in s.

a) Find the average velocity for the interval t=0 to t=2 s

b) Find the instantaneous velocity at t=1.5 s

Solution to find x_i and x_f, we have to substitute for t_i and t_f in the x-t relation, that is,

$$x_i = (t_i)^2 + 3(t_i) = 0$$

and $$x_f = (t_f)^2 + 3(t_f) = 4 + 6 = 10\,\text{m}$$

Now we have for the average velocity $$\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{10 - 0}{2 - 0} = 5\,m/s$$

And for the instantaneous velocity $$v = \frac{dx}{dt} = 2t + 3$$

The instantaneous velocity at t= 1.5 s is obtained by substituting for t= 1.5 s in the last equation

$$v = 2(1.5) + 3 = 6\,m/s$$

2.3 Acceleration

When the velocity of a moving body changes with time, we say that the body has acceleration.

The average acceleration of a particle is defined as

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

The SI unit of velocity is m/s^2.

The instantaneous acceleration of a particle is defined as

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \quad \Rightarrow \quad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

Example 2.2 The velocity of an object varies with time according to $V=5t-3$ with v in m/s and t in s.

a) Find the average acceleration for the interval t=1 s to t=2 s

b) Find the instantaneous acceleration at t=2 s

Solution to find v_i and v_f , we have to substitute for t_i and t_f in the v-t relation, that is,

$$v_i = 5(t_i) - 3 = 5 - 3 = 2\,\text{m/s}$$

$$\text{and } x_f = 5(t_f) - 3 = 10 - 3 = 7\,\text{m/s}$$

Now we have for the average acceleration $\bar{a} = \frac{v_f - v_i}{t_f - t_i} = \frac{7-2}{2-1} = 5\,\text{m/s}^2$

And for the instantaneous acceleration $a = \frac{dv}{dt} = 5\,\text{m/s}^2$

Note that the acceleration is independent of time (constant).

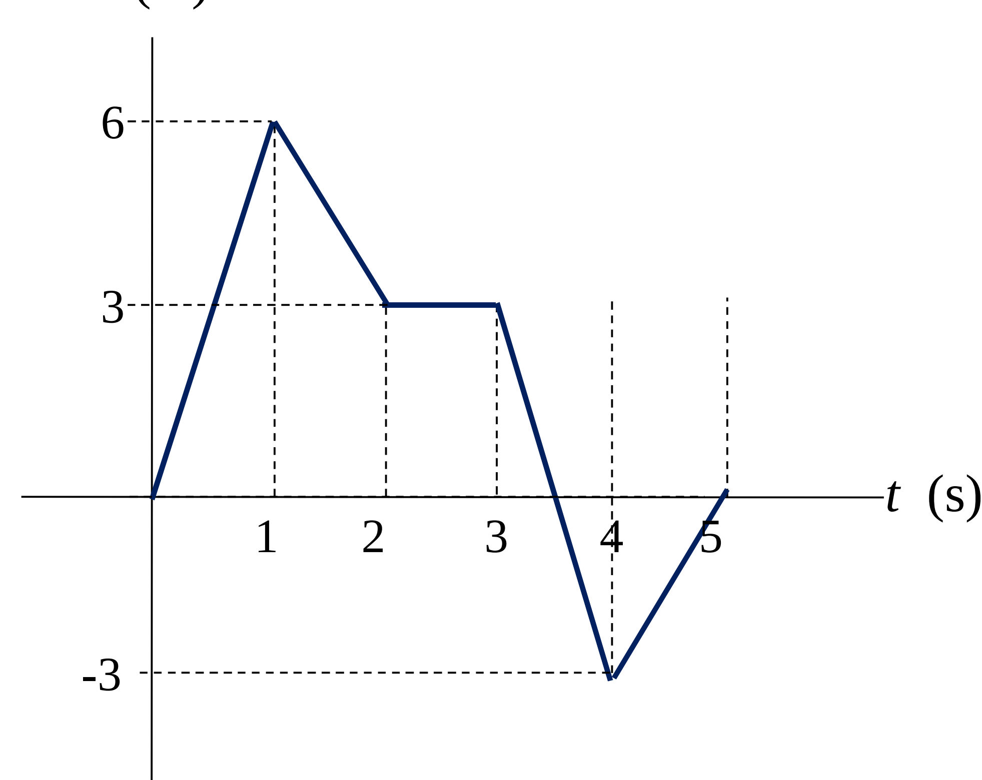
Example 2.3 The position-time graph of a particle moving along the *x*-axis is given in the figure. Find

a) Find the average velocity for the interval t=2 s to t=5 s.

b) Find the average acceleration for the interval t=0.5 s to t=2.5 s.

Solution a) As it is clear from the graph, x_i = 3 m , and x_f = 0. Now we get

$$\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{0-3}{5-2} = -1\,\text{m/s}$$



b) Since *v* at any point is the slope of the *x*-*t* graph at that point, we have at t=0.5 s v=6m/s, and at t=2.5 s v=0, so

$$\bar{a} = \frac{v_f - v_i}{t_f - t_i} = \frac{0-6}{2.5-0.5} = -3\ \text{m/s}^2$$

2.4 Linear motion with constant acceleration

The simplest type of linear motion is the uniform motion in which the acceleration is constant. In such case $\bar{a} = a \quad \Rightarrow$

$$a = \frac{v_f - v_i}{t_f - t_i} = \frac{v - v_o}{t}$$

here we denote v_i by v_o , v_f by v, and t_f by t, and take $t_i = 0$. The above equation now is

$$\boxed{v = v_o + at}$$

Also, because a is constant, we can write

$$\bar{v} = \frac{v + v_o}{2} \quad \Rightarrow \quad \bar{v} = \frac{x - x_o}{t} = \frac{v + v_o}{2} = \frac{v_o + at + v_o}{2} \quad \Rightarrow$$

$$\boxed{x - x_o = v_o t + \frac{1}{2} a t^2}$$

Now as

$$v = v_o + at \Rightarrow t = \frac{v - v_o}{a} \Rightarrow$$

$$x - x_o = v_o\left(\frac{v - v_o}{a}\right) + \tfrac{1}{2}a\left(\frac{v - v_o}{a}\right)^2 = \frac{2vv_o - 2v_o^2 + v^2 + v_o^2 - 2vv_o}{2a}$$

$$x - x_o = \frac{v^2 - v_o^2}{2a} \Rightarrow$$

$$\boxed{v^2 = v_o^2 + 2a(x - x_o)}$$

Strategy for solving problems with constant acceleration:

(i) Choose your coordinates such that the particle begins its motion from the origin ($x_o = 0$).

(ii) Decide the sense of the positive direction.

(iii) Make a list of the known quantities. Do not forget to write any vector quantity (x, v, v_o , a) that have a direction opposite to your positive sense as a negative quantity.

(iv) Make sure that all the quantities have the same system of units.

(iv) (v) According to what is given and what is requested, you can easily decide which equation or equations you need to solve for the unknowns.

Example 2.4 A car starts from rest and moves with constant acceleration. After 12 s its velocity becomes 120 m/*s*. Find,

a) the acceleration of the car

b) the distance the car travels in the 12 s

Solution Let the direction of motion be along the positive *x*-axis, where the car starts from the origin at t=0 (x_o = 0). Now

$$v = v_o + at \Rightarrow a = \frac{v - v_o}{t}$$

But v_o = 0, v=120 m/s and t=12 s. $\Rightarrow$

$$a = \frac{120 - 0}{12} = 10\,\mathrm{m/s^2}$$

To find the distance we use

$$x = v_o t + \tfrac{1}{2} a t^2 = 0 + \tfrac{1}{2}(10)(12)^2 = 720\,\mathrm{m}$$

2.5 Free Falling Bodies

A freely falling body is any body moving freely under the influence of gravity regardless of its initial motion.

Assuming that the gravitational acceleration, denoted by g, is constant, we can consider the motion of a free falling body as a linear motion with constant acceleration.

Since the motion is vertical we take our axis to be the y-axis with the +ve since is upward.

So we replace x by y and the acceleration a by $-g$ Þ

The –ve sign is because the gravitational acceleration is always downward. Þ

$$g=9.8 \text{ m/s}^2 = 32 \text{ ft/s}^2$$

$$v = v_o - gt$$

$$y = v_o t - \frac{1}{2} g t^2$$

$$v^2 = v_o^2 - 2gy$$

Remark: Remember that the gravitational acceleration, *g*, is constant in magnitude and in direction and this means that the negative sign of *g* in the last equations will not be changed unless you change your positive sense, regardless of the direction of motion.

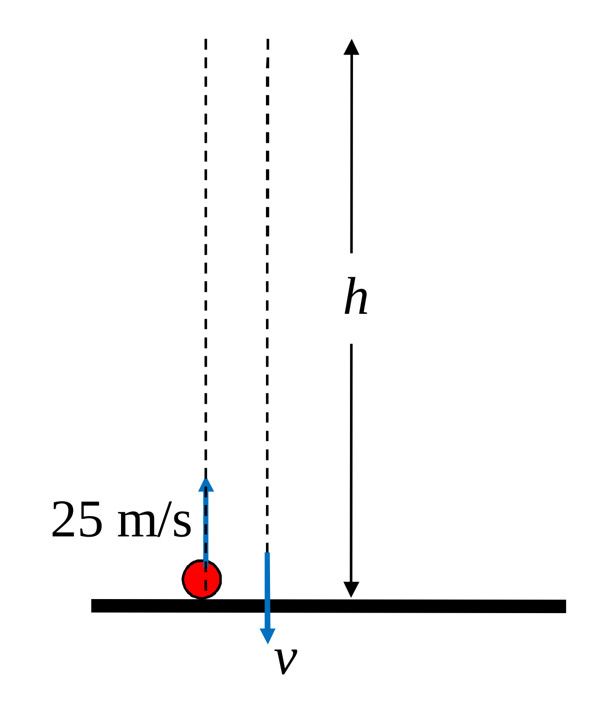
Example 2.4 An object is thrown vertically upward with an initial speed of 25 m/s.

a) How long does it take to reach its maximum height?

b) What is the maximum height?

c) How long does it take to return to the ground?

d) What is its velocity just before striking the ground?



Solution a) At the maximum point v= 0. using the equation

$$v = v_o - gt \qquad \Rightarrow \; t = \frac{v_o - v}{g} \qquad = \frac{25 - 0}{9.8} = 2.55\,\text{s}$$

b) The maximum high is the vertical distance with *v*= 0. using the equation

$$v^2 = v_o^2 - 2gy \;\Rightarrow\; y = \frac{v_o^2 - v^2}{2g} \quad\Rightarrow\quad h = \frac{(25)^2 - 0}{2(9.8)} = 31.9\,\text{m} \quad\Rightarrow$$

c) When returning to the ground, the total displacement of the object is zero (*y* = 0). using the equation

$$y = v_o t - \tfrac{1}{2} g t^2 \Rightarrow 0 = 25t - \tfrac{1}{2}(9.8)t^2 \Rightarrow t = \frac{25}{4.9} = 5.1\text{s}$$

d) To find the final velocity of the motion we use the equation

$$v = v_o - gt = 25 - (9.8)(5.1) = -25\,\text{m/s}$$

The minus sign indicates that the direction of the velocity is downward

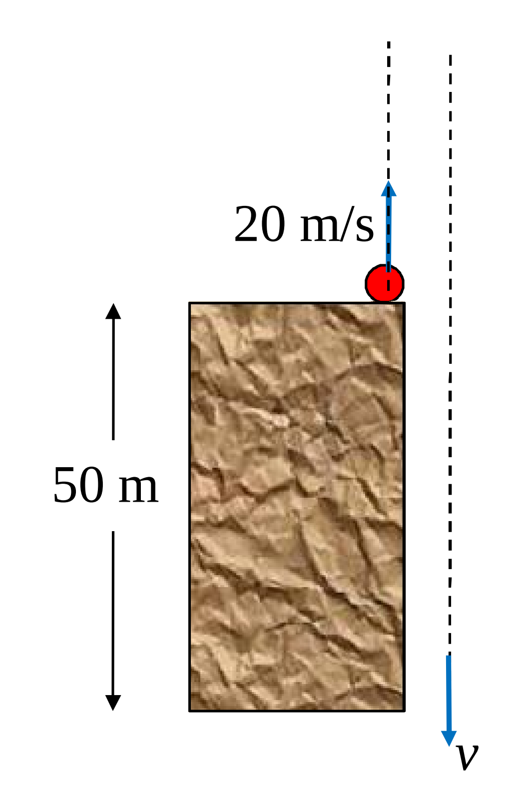
Note that if we use the eq. $v^2 = v_o^2 - 2gy$ with y = 0 we get the same result.

Example 2.6 A student, stands at the edge of the roof of a building, throws a ball vertically upward with an initial speed of 20 m/s. The building is 50 m high, and the ball just missed the edge of the roof in its way down. Find,

a) the time needed for the ball to return to the level of the roof,

b) the velocity and the position of the ball at $t = 5$ s,

c) the velocity of he ball just before hitting the ground.



Solution a) When the ball returns to the level of the roof, its displacement, y, is zero. $\Rightarrow$

$$y = v_o t - \tfrac{1}{2} g t^2 \Rightarrow \quad 0 = 20t - \tfrac{1}{2}(9.8)t^2 \Rightarrow t = \frac{20}{4.9} = 4.08\,\text{s}$$

b) Using the equation

$$v = v_o - gt = 2 - -(9.8)(5) = -29\,\text{m/s}$$

And using the equation

$$y = v_o t - \tfrac{1}{2} g t^2 \Rightarrow \quad y = (20)(5) - \tfrac{1}{2}(9.8)(5)^2 = -22.5\,\text{m}$$

c) From the equation

$$v^2 = v_o^2 - 2gy \qquad \Rightarrow \quad v^2 = (20)^2 - 2(9.8)(-50) \quad \Rightarrow$$

$$v = -37.15\,\text{m/s}$$

The positive solution is rejected because the ball hits the ground while falling.

2.5 RELATIVE MOTION

Suppose that two persons want to observe the motion of a particle *P*. The first person Omer is stationary with respect to the earth, while the second person, Ahmed is moving with constant speed relative to the earth.

Let the reference of frame of Omer be referred by frame *E* (frame of the earth) and the reference of frame of Ahmed by frame *A*.

The displacement of the particle w.r.t. to the earth is denoted by x_{PE},

the displacement of the particle w.r.t. Ahmed is denoted by x_{pA}

while x_{AE} is is the displacement of the *A* relative to *E*.

From the figure we can write

$$x_{PE} = x_{pA} + x_{AE}$$

Differentiate w.r.t. *t* we get

$$v_{PE} = v_{pA} + v_{AE}$$

Example 2.7 A man, in a car, is driving on a straight highway at constant speed of 70 km/h relative to the earth. Suddenly, he spots a truck traveling in the same direction with constant speed of 60 km/h relative to the earth.

a) What is the velocity of the truck relative to the car?

b) What is the velocity of the car relative to the truck?

c) If the car spots the truck when they are 1.5 km apart, how long does it take the car to overtake the truck?

Solution a) Let v_{TC} denotes the velocity of the truck relative to the car, and v_{TE} and v_{CE} denote, respectively, the velocities of the truck and the car relative to the earth. ⇒

$$v_{TE} = v_{TC} + v_{CE} \quad \Rightarrow$$

$$v_{TC} = v_{TE} - v_{CE} = 60 - 70 = -10\,\text{km/h}$$

The minus sign indicates that the car is moving in the opposite direction relative to the car.

b) Again the velocity of the car relative to the truck v_{CT} is

$$v_{CE} = v_{CT} + v_{TE} \quad \Rightarrow$$

$$v_{CT} = v_{CE} - v_{TE} = 70 - 60 = 10\,\text{km/h}$$

Note that as one should expect.

c) The time needed for the car to overpass the truck is

$$t = \frac{1.5}{v_{CT}} = \frac{1.5}{10} = 0.15\,\text{h} = 9\,\text{minutes}$$

CHAPTER 3
PLANAR MOTION

3.1 DISPLACEMENT, VELOCITY, AND ACCELERATION.

Consider a particle moves along the curve shown from initial point *i* to a final point *f*.

The initial and the final position of the particle are two-dimensional vectors:

$$\boldsymbol{r}_i = x_i\boldsymbol{i} + y_i\boldsymbol{j}$$

$$\boldsymbol{r}_f = x_f\boldsymbol{i} + y_f\boldsymbol{j}$$

Now the displacement of the particle is

$$\Delta\boldsymbol{r} = \boldsymbol{r}_f - \boldsymbol{r}_i = (x_f - x_i)\boldsymbol{i} + (y_f - y_i)\boldsymbol{j} \quad = \Delta x\,\boldsymbol{i} + \Delta y\,\boldsymbol{j}$$

Now the average velocity of the particle is

$$\bar{\boldsymbol{v}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} = \bar{v}_{\mathrm{x}}\mathbf{i} + \bar{v}_{\mathrm{y}}\mathbf{j}$$

The instantaneous velocity is now defined as

$$\boldsymbol{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = v_{\mathrm{x}}\mathbf{i} + v_{\mathrm{y}}\mathbf{j}$$

Similarly, the average and the instantaneous acceleration s are now defined as

$$\bar{\boldsymbol{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\mathbf{i} + \frac{\Delta v_y}{\Delta t}\mathbf{j} = \bar{a}_{\mathrm{x}}\mathbf{i} + \bar{a}_{\mathrm{y}}\mathbf{j}$$

$$\boldsymbol{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} = a_{\mathrm{x}}\mathbf{i} + a_{\mathrm{y}}\mathbf{j}$$

It is clear now that the planar motion can be considered as a vector sum of two-perpendicular linear motions: one along the x-axis and the other along the y-axis.

Example 3.1 The coordinates of a particle moving in the *x*-*y* plane are given as a function of time by $\mathbf{r} = 2t\,\mathbf{i}+(19-2t^2)\mathbf{j}$ where **r** in m and *t* in s.

a) What is the average velocity of the particle during the interval $t = 0$ to $t = 2$ s.

b) What is the velocity of the particle at $t = 2$ s?

c) What is the acceleration of the particle at $t = 2$ s?

Solution Let us divide the problem into two parts: one along the x-axis and the other along the y-axis. Now we have:

x-axis	y-axis
$x = 2t$	$y = 19 - 2t^2$
$\bar{v}_x = \dfrac{x_f - x_i}{t_f - t_i} = \dfrac{4-0}{2-0} = 2\,m/s$	$\bar{v}_y = \dfrac{y_f - y_i}{t_f - t_i} = \dfrac{11-19}{2-0} = -4\,m/s$
$\bar{\mathbf{v}} = (2\mathbf{i} - 4\mathbf{j})$	

b) For the instantaneous velocity we have

x-axis	y-axis
$x = 2t$	$y = 19 - 2t^2$
$v_x = \dfrac{dx}{dt} = 2\ \mathrm{m/s}$	$v_y = \dfrac{dy}{dt} = -4t = -8\ \mathrm{m/s}$
$\boldsymbol{v} = (2\mathbf{i} - 8\mathbf{j})\ \mathrm{m/s}$	

b) For the instantaneous acceleration we have

x-axis	y-axis
$x = 2t$	$y = 19 - 2t^2$
$a_x = \dfrac{d^2x}{dt^2} = 0$	$a_y = \dfrac{d^2y}{dt^2} = -4\ \mathrm{m/s}$
$\boldsymbol{a} = (-4\mathbf{j})\ \mathrm{m/s}$	

If **a** is constant, then a_x and a_y are consequently constants, then we can write, by letting $x_o = y_o = 0$,

x-axis	y-axis
$v_x = v_{ox} + a_x t$	$v_y = v_{oy} + a_y t$
$x = v_{ox} t + \frac{1}{2} a_x t^2$	$y = v_{oy} t + \frac{1}{2} a_y t^2$
$v_x^2 = v_{ox}^2 + 2 a_x x$	$v_y^2 = v_{oy}^2 + 2 a_y y$

3.2 PROJECTILE MOTION

If a particle given an initial velocity and then follows a path determined only by gravity we have projectile motion.

As the acceleration of gravity is always straight downward with a magnitude of *g*, the acceleration of a projectile is always given as

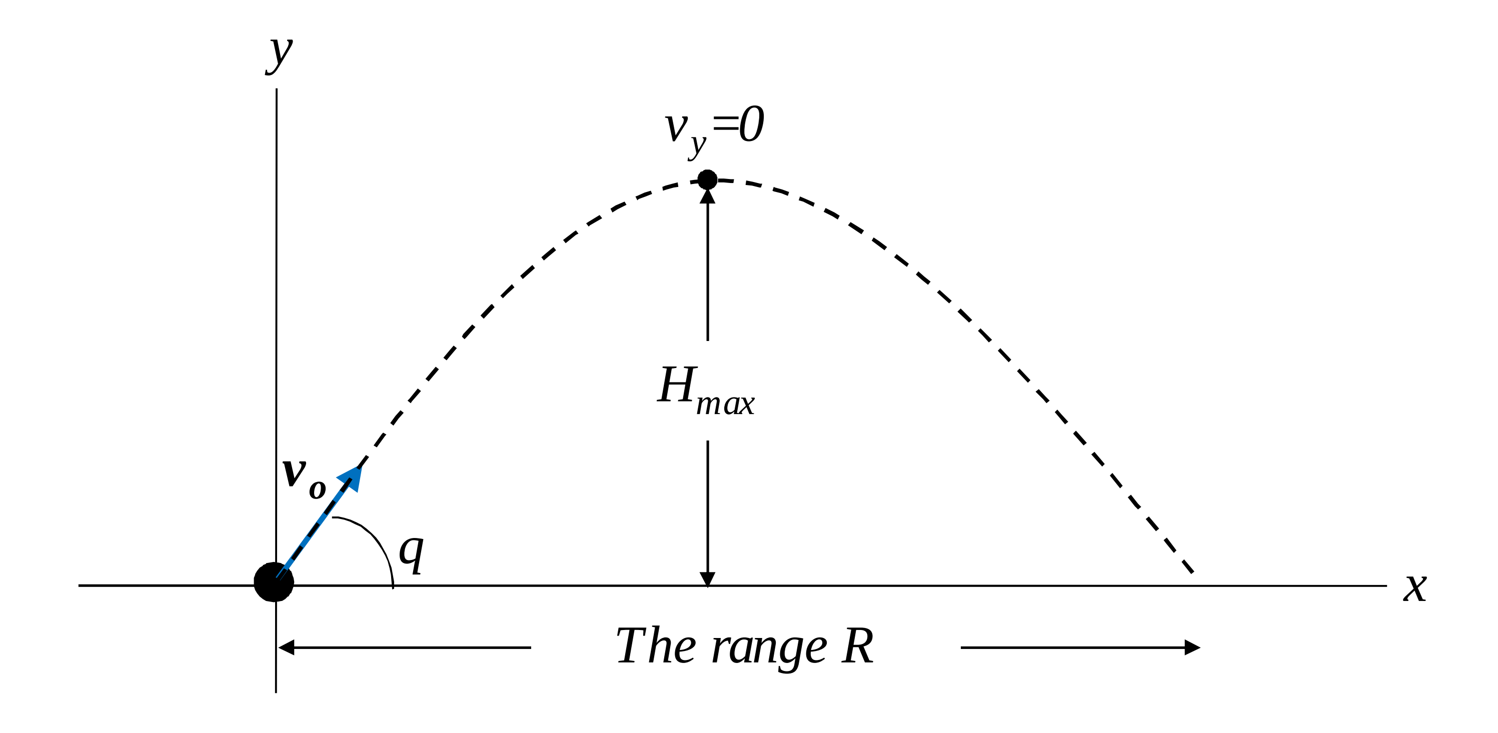
$$\boldsymbol{a} = -g\mathbf{j}$$

This means that the projectile motion can be considered as a vector sum of a vertical motion with $a_y = -g$ (the positive sense is taken upward) and a horizontal motion with $a_x = 0$ (constant velocity). The formulas govern the motion of a projectile is then

Horizontal motion ($a_x = 0$)	Vertical motion ($a_y = -g$)
$v_x = v_{ox}$	$v_y = v_{oy} - gt$
$x = v_{ox}t$	$y = v_{oy}t - \frac{1}{2}gt^2$
$v_x^2 = v_{ox}^2$	$v_y^2 = v_{oy}^2 - 2gy$

If one substitute for *t* from *x*-eq. into the 2nd y-eq. we get the equation of a parabola.

$$y = (v_o \tan q)x - \left(\frac{g}{2v_o^2 \cos^2 q}\right)x^2$$



Example 3.2 A projectile is fired from point *O* with initial speed *v*o that make an angle with the horizontal as shown in Fig 3.2.

a) Find the maximum height *h* of the projectile.

b)Find the horizontal range *R* of the projectile.

Solution Resolving the initial velocity *v* into its components we get

$$v_{ox} = v_o \cos q \qquad \text{and}\, v_{oy} = v_o \sin q$$

a)At the maximum height (h), v_y =0, using the equation

$$v_y^2 = v_{oy}^2 - 2gy \qquad \Rightarrow$$

$$0 \ = (v_o \sin q)^2 - 2gh. \qquad \Rightarrow$$

$$h = \frac{v_0^2 \sin^2 q}{2g}$$

b) To find the range we need the time of flight. Noting that for the total time of flight *y*=0, so we can use

$$y = v_{\text{oy}} t - \frac{1}{2} g t^2 \quad \Rightarrow \quad t = \frac{2v_{oy}}{g} = \frac{2v_o \sin q}{g} \quad \Rightarrow$$

Now substituting for *t* in the eq. $x = v_{\text{ox}} t \quad \Rightarrow$

$$R = v_{\text{ox}} \frac{2v_o \sin q}{g} = \frac{2v_o^2 \sin q \cos q}{g}$$

Knowing that $2 \sin q \cos q = \sin 2q \quad \Rightarrow$

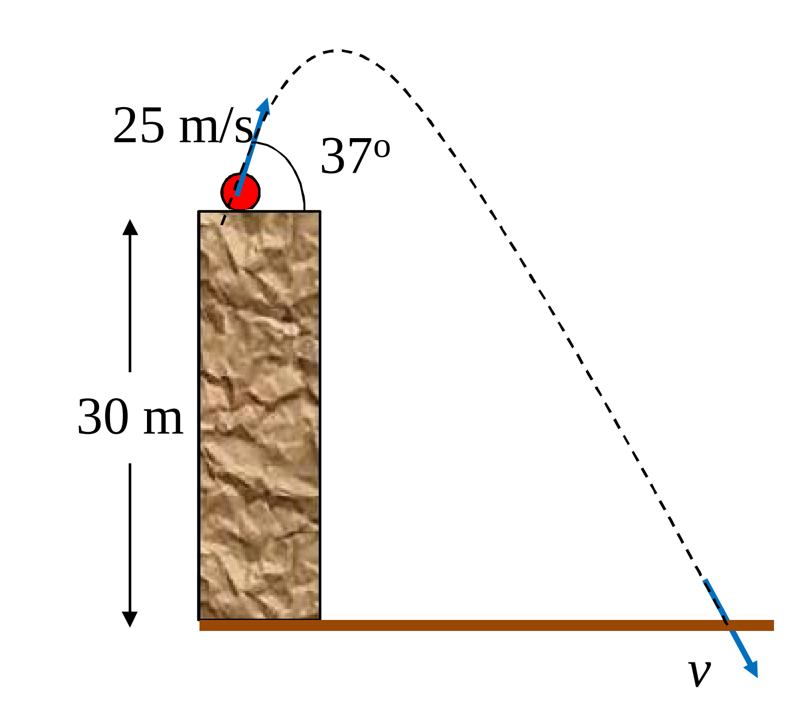
$$R = \frac{v_o^2 \sin 2q}{g}$$

Example 3.3 A ball is thrown from the top of a building 30 m in height. If the ball is thrown upward with a speed of 25 m/s that making an angle of 37o with the horizontal.

a)What is the time of flight?

b)Where does the ball hit the ground?

c)What is the speed of the ball just before it hits the ground?



Solution Resolving the initial velocity into its components we get

$v_{ox} = v_o \cos q = 25 \cos 37 = 20\,\text{m/s}$ and $v_{oy} = v_o \sin q = 25 \sin 37 = 15\,\text{m/s}$

b) To find t we use $y = v_{oy} t - \frac{1}{2} g t^2 \quad \Rightarrow$

$$-30 = 15t - 4.9t^2 \quad \Rightarrow$$

why

$$t = 4.44\,\text{m/s}$$

b) It is enough to find the horizontal distance, we have

$$x = v_{ox} t = (20)(4.44) = 88.8\,\text{m} \;\Rightarrow$$

The ball hits the ground 88.8 m from the base of the building.

c) To find the final speed, we have to find the two components of the final velocity. Using

$$v_y = v_{oy} - gt \quad \Rightarrow \quad v_y = 15 - (9.8)(4.44) = -28.5\,\text{m/s} \;\Rightarrow$$

Noting that the horizontal velocity is constant we can write

$$\boldsymbol{v} = v_x \mathbf{i} + v_y \mathbf{j} = 20\mathbf{i} - 28.5\mathbf{j}\,\text{m/s}$$

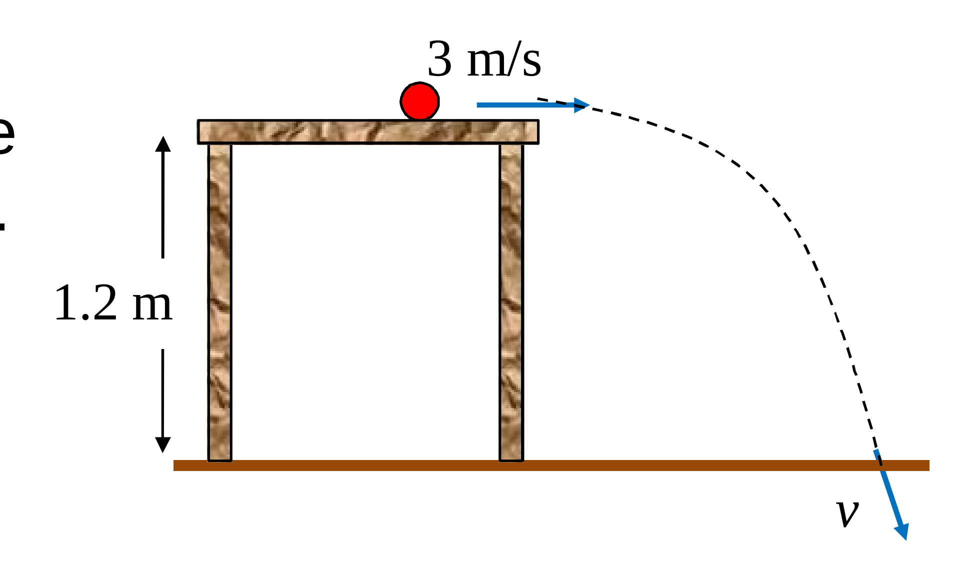
And for the speed we have

$$v = \sqrt{v_x^2 + v_y^2} = 34.8\,\text{m/s}$$

Example 3.4 A ball rolls off the edge of a tabletop 1.2m above the floor with an initial speed of 3 m/s.

a)Where does the ball strike the floor?

b)What is the final velocity of the ball?



Solution It clear now that there is no vertical component for v_o Þ

$$v_{ox} = 3\,\text{m/s} \qquad \text{and}\; v_{oy} = 0$$

To find the horizontal dis why e need the time. Using

$$y = v_{oy}t - \frac{1}{2}gt^2 \quad \Rightarrow \quad -1.2 = -4.9t^2 \quad \Rightarrow \quad t = 0.49\,\text{s} \quad \Rightarrow$$

$$x = v_{ox}t = (3)(0.49) = 1.48\,\text{m} \quad \Rightarrow$$

b) To find the final speed, we again have to find the two components of the final velocity. Using

Down

$$v_y = v_{oy} - gt \quad \Rightarrow \quad v_y = 3 - (9.8)(0.49) = -4.8\,\text{m/s} \quad \Rightarrow$$

Noting that the horizontal velocity is constant we can write

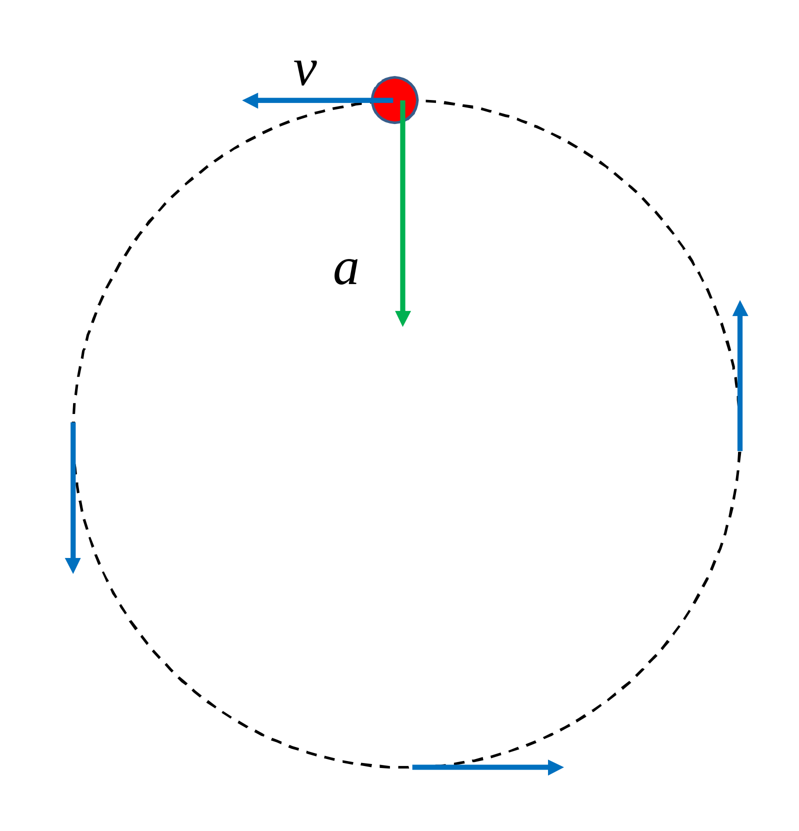
$$\boldsymbol{v} = v_x\mathbf{i} + v_y\mathbf{j} = 3.0\mathbf{i} - 4.8\mathbf{j}\ \text{m/s}$$

3.3 UNIFORM CIRCULAR MOTION

When an object moves on a circular path with constant speed *v*, the object is said to be in a uniform circular motion.

As the velocity, in a uniform circular motion, changes continuously in direction,

and although the speed is constant, the velocity as a vector quantity is changeable resulting in acceleration directed toward the center (**centripetal acceleration**). and is given as



$$a = \frac{v^2}{r}$$

Example 3.5 The moon revolves about the earth in an orbit (assuming it is a circular) of radius 3.85×10^5 km and makes one revolution in 27.3 days. Find the acceleration of the moon toward the earth.

Solution The time for one revolution, called the period is

$$T = 27.3 \times 24 \times 3600 = 2.36 \times 10^6 \text{ s.}$$

The speed of the moon is now

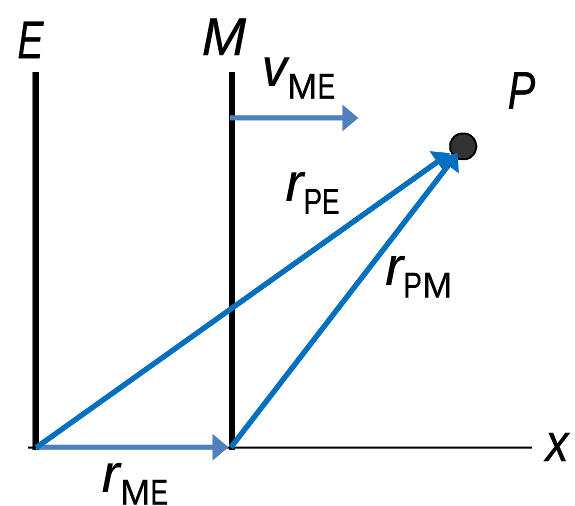
$$v = \frac{2pr}{T} = \frac{2p\left(3.85 \times 10^8\right)}{2.36 \times 10^6} = 1026 \text{ m/s}$$

The centripetal acceleration is now

$$a = \frac{v^2}{r} = \frac{(1020)^2}{3.85x10^8} = 2.73 \times 10^{-3} \text{m/s}^2$$

3.4 RELATIVE MOTION

Suppose that two persons want to observe the motion of a particle *P*. The first person Ahmed is stationary with respect to the earth, while the second person, Mustafa in frame *M*, is moving with constant velocity $\boldsymbol{v}_{ME}$ relative to the earth.



If the two reference frames coincide at $t=0$, then after a time *t*, the displacement of *M* relative to *E* will be $\boldsymbol{r}_{ME}.= \boldsymbol{v}_{ME}\, t$

The displacement of the particle w.r.t. to the earth is denoted by $\boldsymbol{r}_{PE}$,

the displacement of the particle w.r.t. Ahmed is denoted by $\boldsymbol{r}_{PM}$

From the figure we can write $\boldsymbol{r}_{PE} = \boldsymbol{r}_{PM} + \boldsymbol{r}_{ME} \qquad = \boldsymbol{r}_{PM} + \boldsymbol{v}_{ME}t$

Differentiate w.r.t. *t* we get $\boldsymbol{v}_{PE} = \boldsymbol{v}_{PM} + \boldsymbol{v}_{ME}$

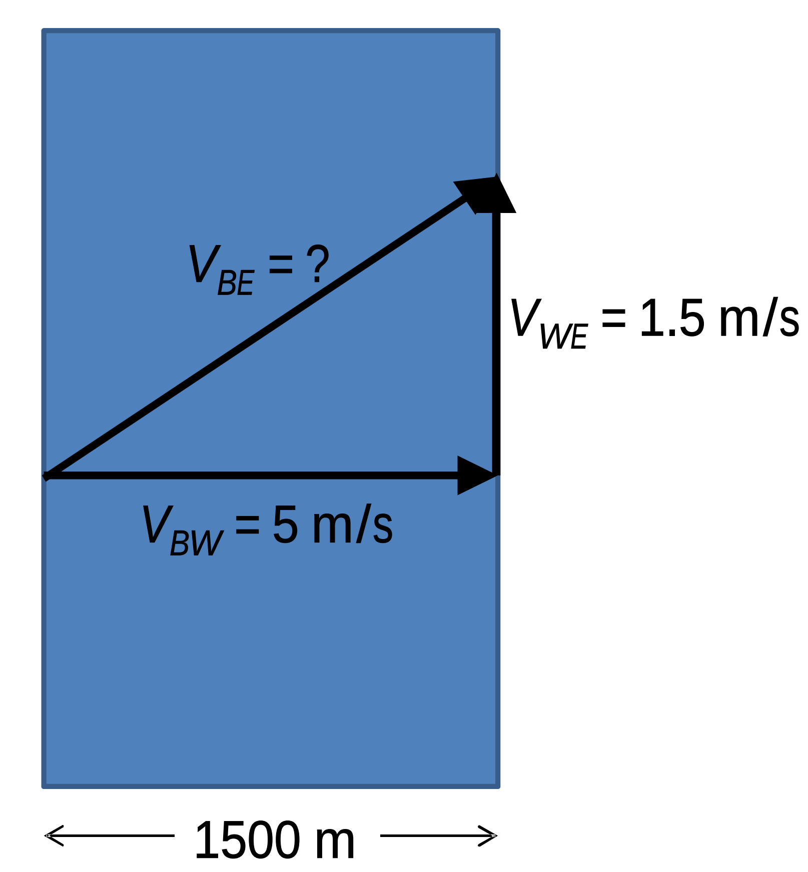
Differentiate again w.r.t. *t* we get $\boldsymbol{a}_{PE} = \boldsymbol{a}_{PM}$

Example 3.6 A man, by a boat, wants to cross a river, 1500 m in wide, and flows due north with a speed of 1.5 m/s. The boat is rowed with a speed of 5 m/s due east relative to the water

a) What is the velocity of the boat relative to the ground?

b) Find the time needed for the boat to cross the river.

c) Determine the point the boat will reach on the opposite bank of the river.



Solution Let $\boldsymbol{v}_{\text{WE}}$, $\boldsymbol{v}_{\text{BE}}$, and $\boldsymbol{v}_{\text{BW}}$ be the velocity of the water relative to the earth, the velocity of the boat relative to the earth, and the velocity of the boat relative to the water, respectively.

$$\boldsymbol{v}_{BE} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WE} \quad \Rightarrow$$

$$\boldsymbol{v}_{BE} = 5\mathbf{i} + 1.5\mathbf{j} \quad \Rightarrow$$

b) The wide of the river can be considered as the horizontal displacement of the boat., i.e. *x=1500* m, so

$$t = \frac{x}{(v_{\mathrm{BE}})_x} = \frac{1500}{5} = 300\,\mathrm{s}$$

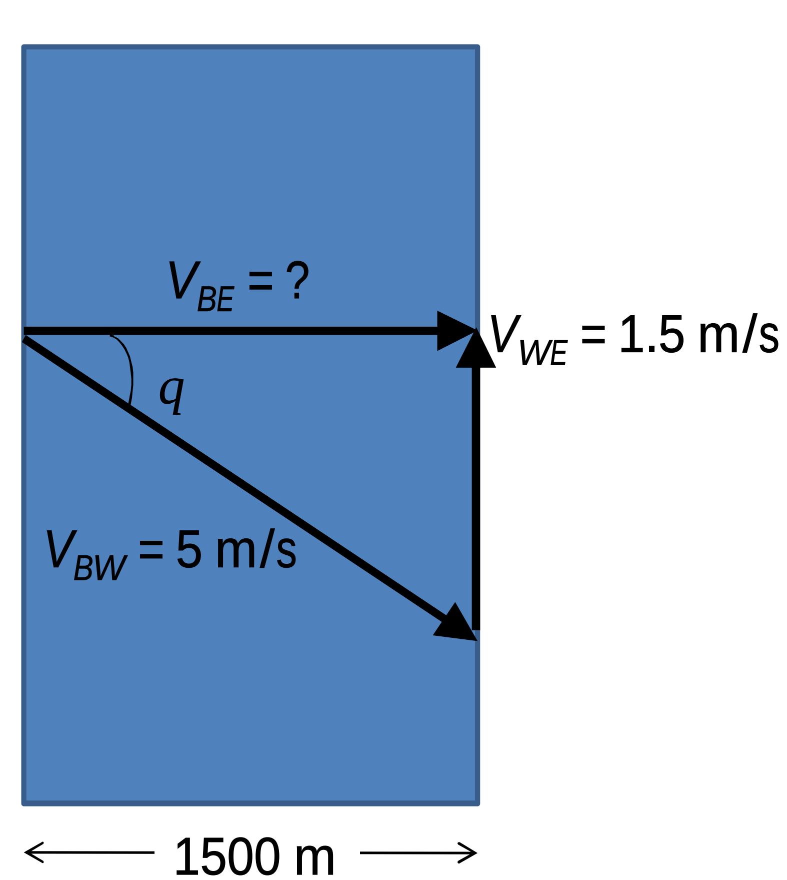
c) Now we want to find the vertical displacement of the boat relative to the earth

$$y = (v_{\mathrm{BE}})_y t = 1.5(300) = 450\,\mathrm{m}$$

i.e. the point is 450 m north of its starting point.

Example 3.7 In the previous example, if the boat have to reach a point on the opposite bank directly east from the starting point,
a) in what direction should the boat be headed.
b) what will be the velocity of the boat relative to the ground?
c) how much time is required to cross the river?

Solution a) For the boat to reach a point directly east from the starting point, the vertical displacement should be zero, i.e. y=0 and this implies that



$$\mathrm{v}_{\mathrm{BE}} = v\mathrm{i}$$

$$\boldsymbol{v}_{BE} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WE} \quad \Rightarrow$$

$$v\mathrm{i} = \mathrm{v}_{BW} + 1.5\mathbf{j} \quad \Rightarrow$$

But we have

$$v_{BW} = \sqrt{v^2 + (-1.5)^2} = 5\,\text{m/s} \quad \Rightarrow$$

$$v = \sqrt{25 - 2.25} = 4.77\,\text{m/s} \;\Rightarrow\; v_{BW} = 4.77\mathrm{i} - 1.5\mathbf{j}$$

$$\tan q = \frac{1.5}{4.77} \Rightarrow q = 17.46^{\text{o}}$$

b) It is clear now that $v_{\text{BE}} = v\mathrm{i} = 4.77\,\mathrm{i}\,\text{m/s}$

c) To find the time of crossing we have

$$t = \frac{x}{(v_{\text{BE}})_x} = \frac{1500}{4.77} = 314.5\,\text{s}$$

CHAPTER 4
NEWTON'S LAWS OF MOTION

4.1 FORCE

When a body affects a second body, we say that the first body exerts a force on the second.

This means that only force can cause the body to change its state of uniform motion or rest.

The force may be contact force or field force.

Contact force is the force that result from the physical contact between the objects, e.g., frictional force

Field force is the force that doesn’t require physical contact between the objects, e.g., gravitational force

The force is a vector quantity with the SI unit of Newton (N). The cgs unit is dyne and its British unit is pound (lb).

4.2 Newton’s First Law

An object continues in its state of rest or uniform motion until it is forced to change that state by an external force.

The frame in which Newton's first law is valid is called **inertial frame**

All accelerated frames are not inertial frames.

The ground can be considered as an approximate inertial frame regardless of its rotation about the sun and about its own axis.

4.3 Mass and Inertia

Inertia is the property of matter that resists the change of its state and mass is the measure of this inertia.

The mass is a scalar quantity with a unit of kilogram (kg), in the SI unit system and gram (g) in the *cgs* unit system.

Remark There is a difference between mass and weight. While the mass of a body is the measure of inertia of that body (the same everywhere), its weight is the force exerted by gravity on it (depends on the body's position).

4.3 Newton's Second Law

The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass, i.e.,

$$\sum \boldsymbol{F} = m\boldsymbol{a}$$

The net force acting on the body.

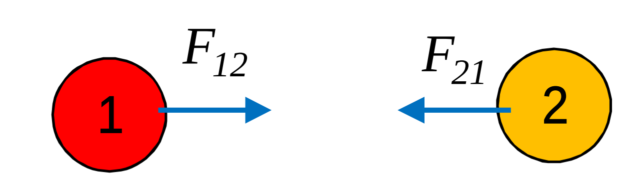
The last eq. can be written as.

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

4.4 Newton's Third Law

for every action there is an equal, but opposite reaction.

if two bodies interact, the force exerted by body number 1 on body number 2 ($\boldsymbol{F}_{21}$) is equal and opposite to the force exerted on body number 1 by body number 2 $\boldsymbol{F}_{12}$ Þ



$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$$

Remark Action and reaction act on different bodies.

Free Body Diagram: It is a diagram showing all the forces acting on the body and do not include the forces the body exert on other bodies.

Strategy for solving problems using Newton’s laws:

(i) Chose a suitable coordinate system with the positive direction is the direction of the acceleration, if it is known.

(ii) Draw a free-body diagram of each body of the system separately.

(iii) Resolve each force into its components according to the chosen coordinates.

(iv) Identify the known and the unknown quantities.

(v) Now you can apply Newton's second law for one body or more of the system according to the unknown quantities.

Example 4.1 A boy want to drag a box, that has a mass of 3 kg, along a horizontal smooth surface. He pulls the box horizontally with a force of 2 N.

Find the acceleration of the box.

Solution Let us first draw the free-body diagram of the system.

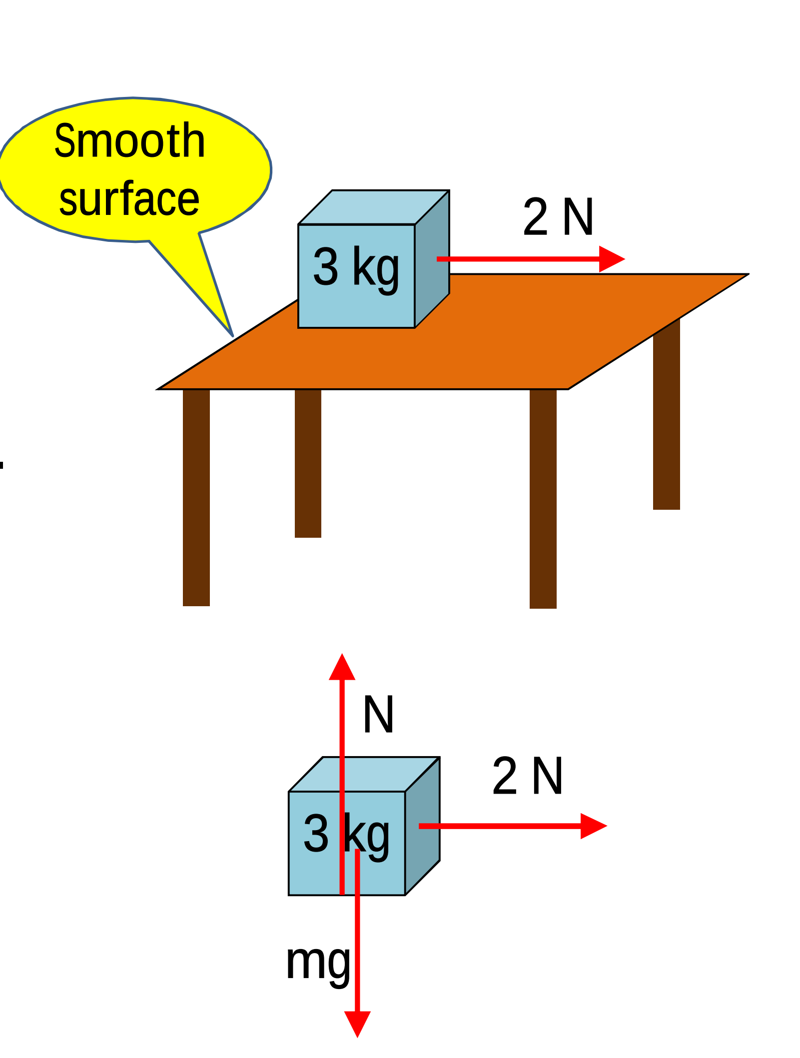
N is called the normal force, which is a force exerted on the box by the surface and *mg* represents the weight of the box.

Now using Newton's 2nd law we get.

$$\sum F_x = ma_x \quad \Rightarrow$$

$$2 = 3a_x = 3a \quad \Rightarrow$$

$$a = \frac{2}{3} = 0.67\,\text{m/s}^2$$



Example 4.2 Two masses m_1 and m_2 ($m_1 > m_2$) are suspended vertically by a light string that passes over a light, frictionless pulley (Atwood's machine). Find the acceleration of the masses and the tension in the string.

Solution Again let us first draw the free-body diagram of the system.

Taking the +ve sense to be downward and using

$$\sum F_y = ma_y \quad \Rightarrow$$

For m_1 : $\quad m_1 g - T = m_1 a \qquad (1)$

For m_2 : $\quad m_2 g - T = -m_2 a \qquad (2)$

Subtracting the 2-eqs. $\Rightarrow$

$$(m_1 - m_2)g = (m_1 + m_2)a \qquad \Rightarrow$$

$$a = \frac{(m_1 - m_2)g}{(m_1 + m_2)}$$

And for *T* we have

$$a = \frac{(2m_1 m_2)g}{(m_1 + m_2)}$$

Example 4.3 Two blocks are in contact on a smooth horizontal table. A constant force *F* is applied to one block as shown.
a) Find the acceleration of the system.
b) Find the contact force between the two blocks

Solution Let us first draw the free-body diagram of the system.

Smooth surface
m_2 m_1 F

Note that $\boldsymbol{F}_C$ represents the contact force due to the contact between the two blocks. They are action and reaction forces.

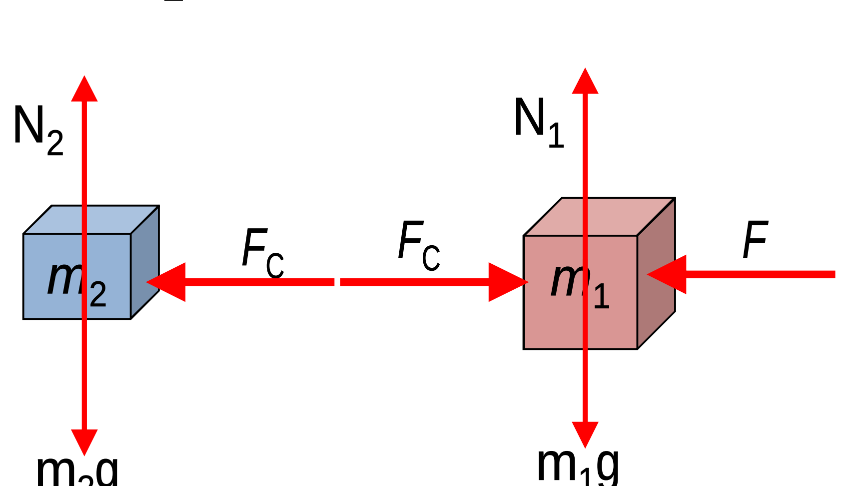
Now using Newton's 2nd law we get for m_1 .

$$\sum F_x = ma_x \quad \Rightarrow$$

$$F - F_C = m_1 a \qquad (1)$$

and for m_2 we get.

$$F_C = m_2 a \qquad (2)$$



Adding the 2-eqs. $\Rightarrow$

$$F = (m_1 + m_2)a \qquad \Rightarrow$$

$$a = \frac{F}{(m_1 + m_2)}$$

Substituting for a in eq. (2) $\Rightarrow$

$$F_C = \frac{m_2 F}{(m_1 + m_2)}$$

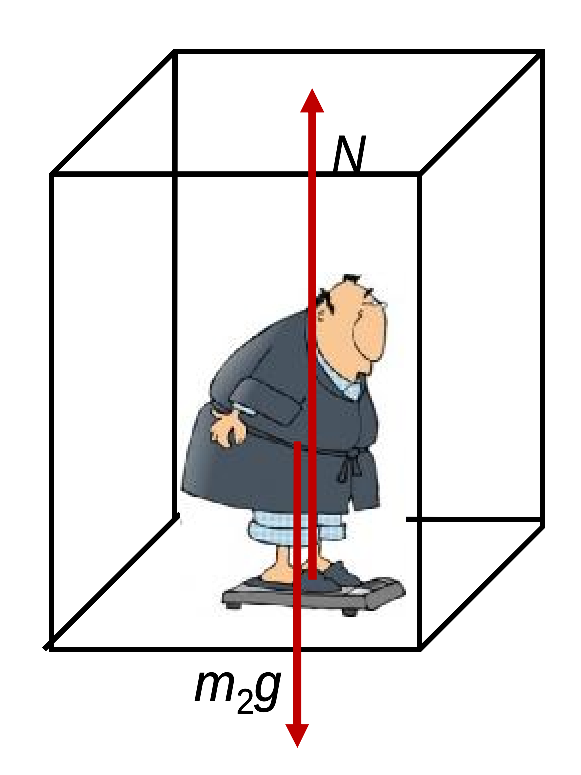
Example 4.4 A man of mass 80 kg stands on a platform scale in an elevator. Find the scale reading when the elevator

a) moves with constant velocity,

b) ascends with an acceleration of 3 m/s^2 ,

c) descends with an acceleration of 3 m/s^2

Solution The free body diagram of the man is



Noting that the scale reads its reaction force, and applying Newton's second law in the three cases, you get

(a) $\sum F_y = ma \quad \Rightarrow$

$N - mg = 0 \quad \Rightarrow N = mg = 784\,\mathrm{N}$

(b) $N - mg = ma \quad \Rightarrow N = m(g + a) = 1024\,\mathrm{N}$

(c) $N - mg = m(-a) \quad \Rightarrow N = m(g - a) = 544\,\mathrm{N}$

What happen if a=g?

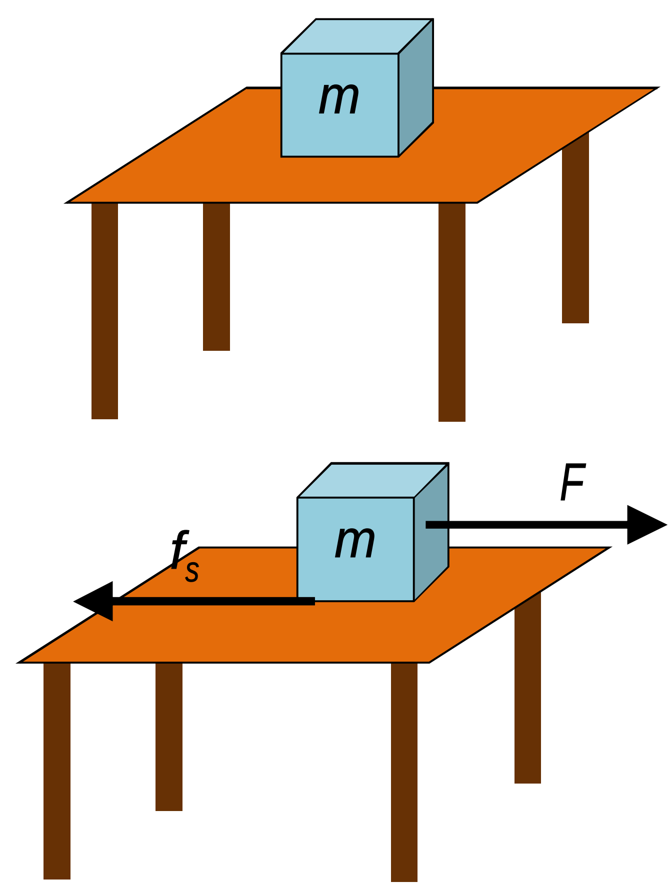
4.6 FRICTIONAL FORCES

The frictional forces is the forces that two surfaces in contact exert on each other to oppose the sliding of one surface over the other.

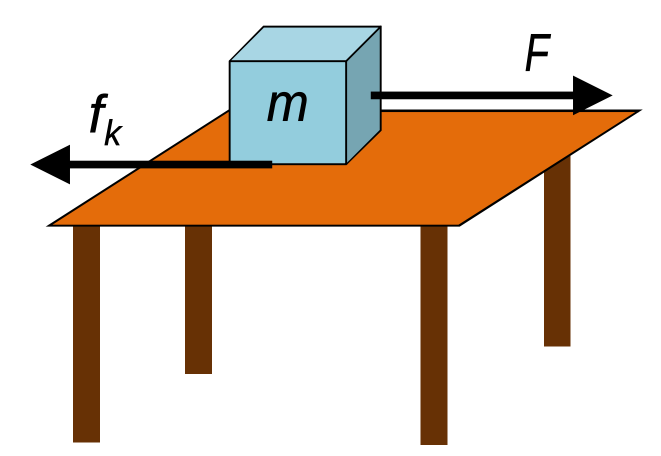
If you want to push you're a block along a horizontal table three cases have to be considered:

(i) No external force is applied. Since a=0 $\Rightarrow$ The frictional force is zero.

(ii) An external force is applied but no motion. Since a=0 $\Rightarrow$ $f_s = F$. This frictional force is called the static frictional force.



(ii) A larger external force is applied $\{F > f_s(max)\}$ so that the block is moving. Since $a \neq 0 \Rightarrow F - f_k = ma$. This frictional force is called the kinetic frictional force.



It is found, experimentally, that the frictional forces f_s and f_k, between two surfaces, are proportional to the normal force N pressing the two surfaces together. i.e.,

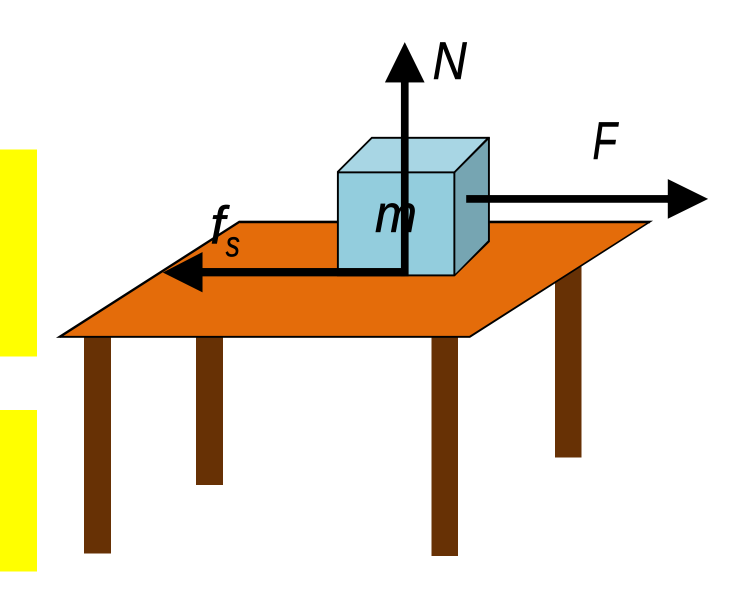
$$f_s \leq m_s N$$

$$f_k = m_k N$$

Where the dimensionless constants m_s and m_k are, respectively, the coefficient of static friction and the coefficient of kinetic friction. It is found that $m_s > m_k$

Ramarks:

(i) The frictional force is always parallel to the surfaces in contact.



(ii) The force of static-friction is always opposite to the applied force.

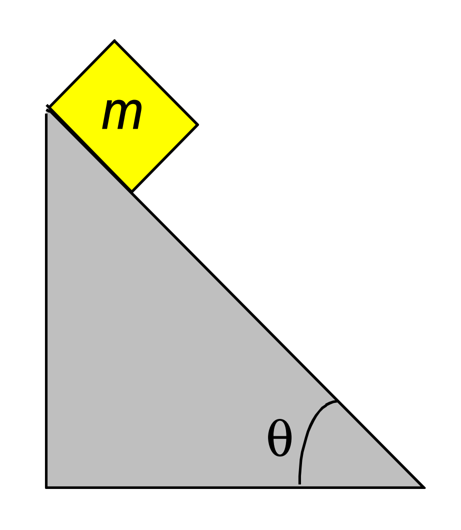
(iii) The force of kinetic friction is always opposite to the direction of motion.

(iv) The frictional force, together with the normal force constitute the two perpendicular components of the reaction force exerted by one of the contact bodies on the other.

Example 4.5 A block of mass *m* slides down a rough, inclined plane with the angle of inclination is as shown. The coefficient of kinetic friction between the block and the plane is m.

a) Find the acceleration of the block.

b) If the block starts from rest at the top of the plane, find its velocity after it slides a distance *d* along the plane.



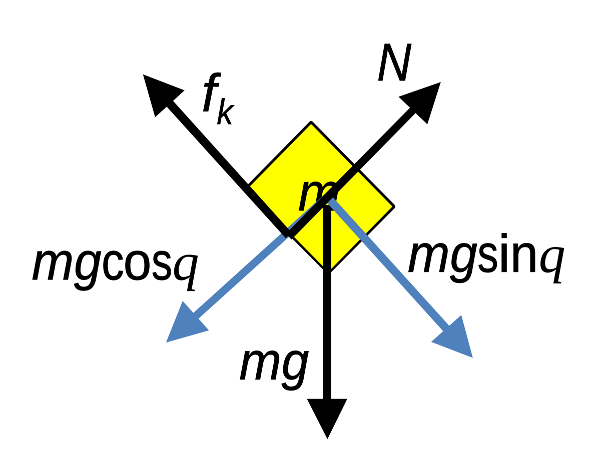
Solution The free body diagram of the block is shown

Taking the x-axis along the motion we have to resolve the weight into its components:

Now applying Newton’s second law in the *y*-axis, we get

$$\sum F_y = N - mg\cos q = 0 \quad \Rightarrow$$

$$N = mg\cos q$$



And in the *x*-axis, we get

$$\sum F_{\mathrm{x}} = mg\sin q - f_{\mathrm{k}} = ma$$

Substituting for f_k by $\quad f_{\mathrm{k}} = m_k N = m_k mg\cos q \quad \Rightarrow$

$$mg\sin q - mgm\cos q = ma \qquad \Rightarrow$$

$$a = g(\sin q - m\cos q)$$

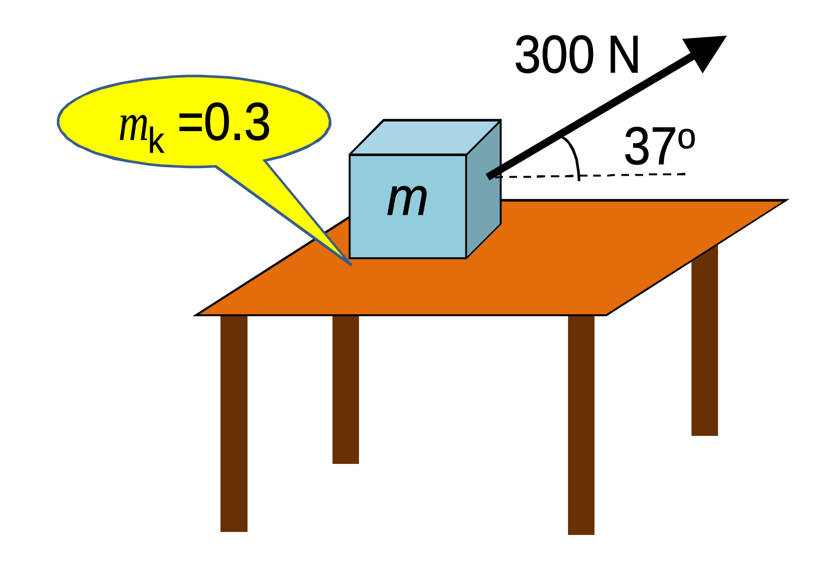
b) Since the acceleration is constant, we use

$$v^2 = v_{\mathrm{o}}^2 + 2ax \qquad \Rightarrow$$

$$v^2 = 0 + 2gd(\sin q - m\cos q) \quad \Rightarrow$$

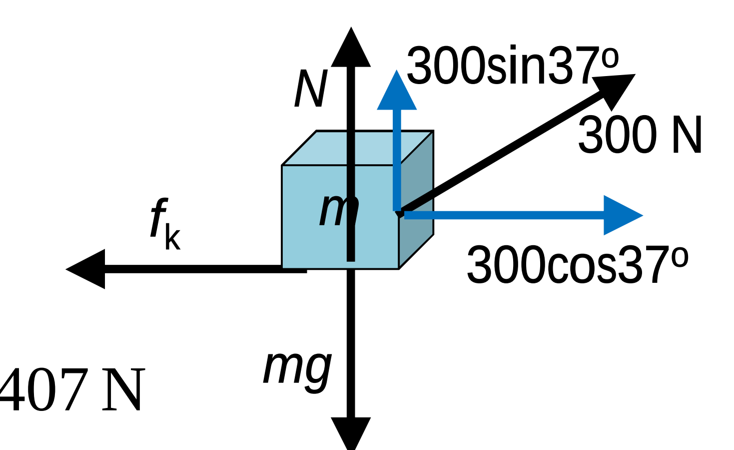
$$v = \sqrt{2gd(\sin q - m\cos q)}$$

Example 4.6 A worker drags a crate along a rough, horizontal surface by pulling on a rope tied to the crate. The worker exerts a force of 300 N on the rope that is inclined 37o to the horizontal as shown. If the mass of the crate is 60 kg, and the coefficient of kinetic friction between the crate and the surface is 0.3, find the acceleration of the crate.



Solution The free body diagram of the block is shown with the normal axis.

Now applying Newton's second law in the *y*-axis, we get



$$\sum F_{\mathrm{y}} = N + F\sin 37^{o} - mg = 0 \quad \Rightarrow$$

$$N = mg - F\sin q = (60)(9.8) - 300\cos 37^{o} = 407\ \mathrm{N}$$

And in the *x*-axis, we get

$$\sum F_{\mathrm{x}} = F\cos q - f_{\mathrm{k}} = ma \quad \Rightarrow$$

Substituting for f_k by $\quad f_{\mathrm{k}} = m_k N = 0.3(407) = 122\ \mathrm{N} \Rightarrow$

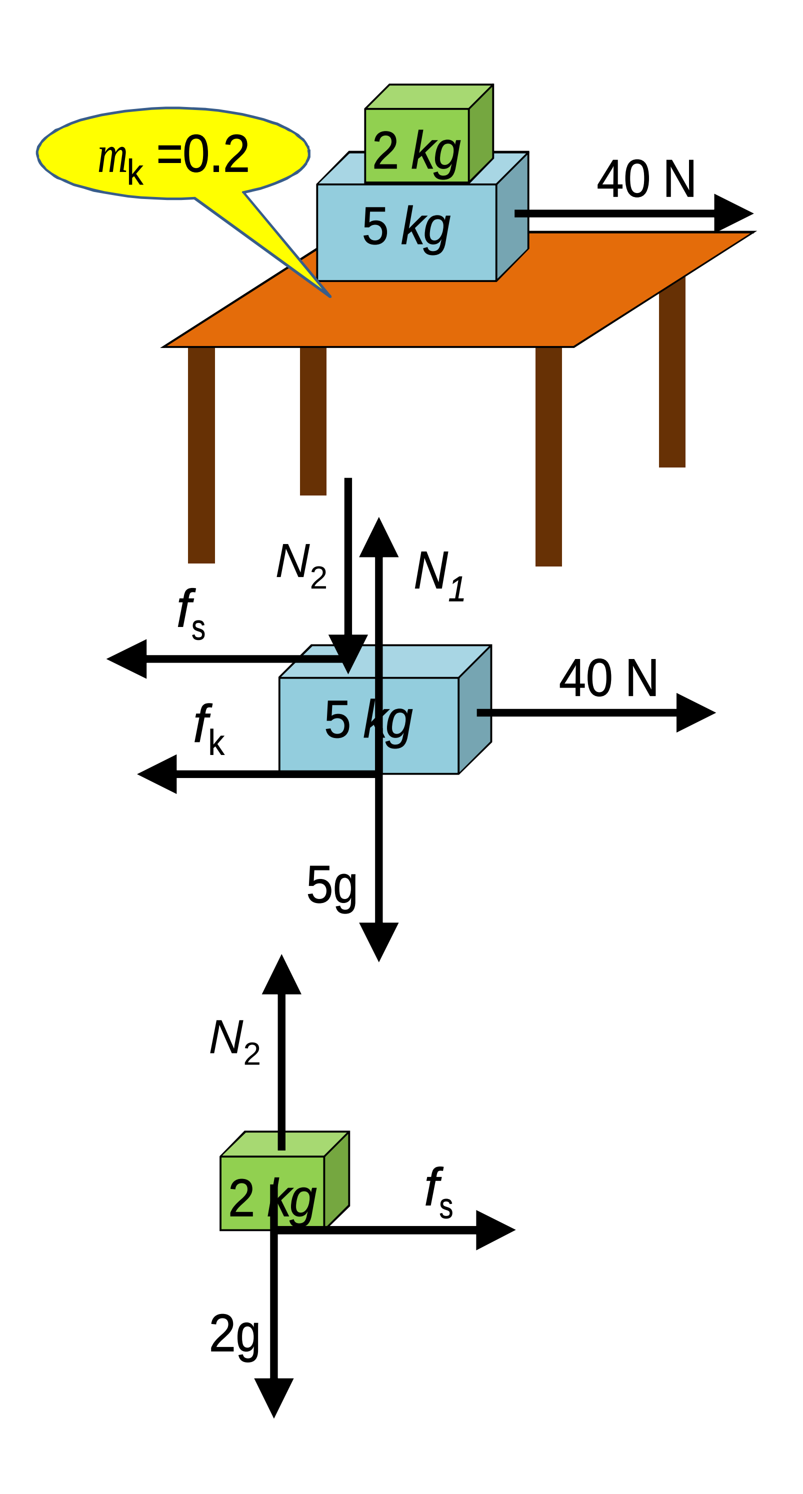
$$300\cos 37^{o} - 122 = 60a \quad \Rightarrow \quad a = 1.96\ \mathrm{m/s}^2$$

Example 4.7 A 2-kg block is placed on top of a 5-kg block as shown. A horizontal force of 40 N is applied to the 5-kg block. If the coefficient of kinetic friction between the 5-kg block and the surface is 0.2, and assuming that the 2-kg block is in the verge of slipping,

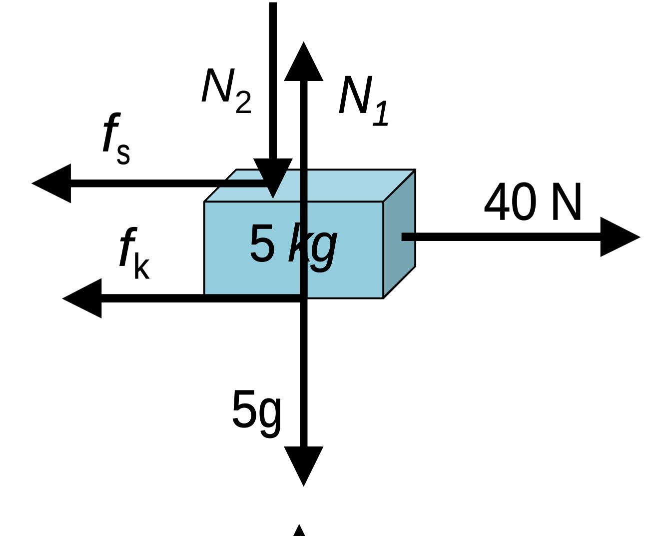
a) what is the acceleration of the system?

b) What is the coefficient of static-friction between the two blocks?

Solution The free body diagram of the two blocks are shown with the normal axis.



Applying Newton's 2nd law to the 2-kg block, we have in the x-axis

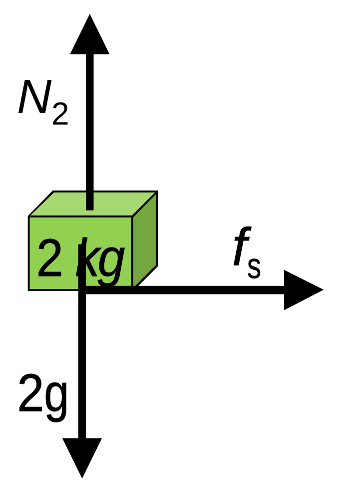


$$f_s = 2a \qquad (1)$$

And in the y-axis $N_2 - 2g = 0 \quad \Rightarrow$

$$N_2 = 19.6\,\text{N} \qquad (2)$$

Similarly for the 5-kg block, we have

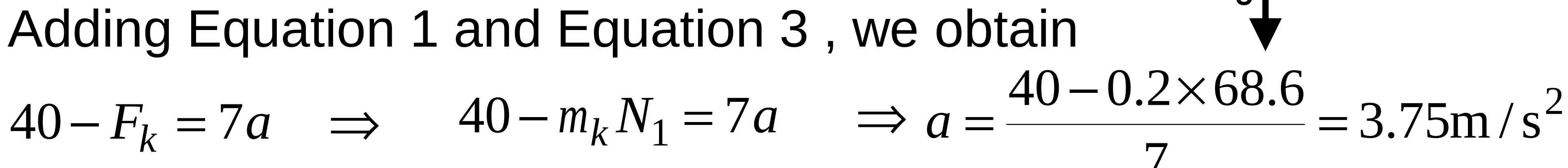


$$40 - F_k - f_s = 5a \qquad (3)$$

$$N_1 - N_2 - 5g = 0 \quad \Rightarrow$$

$$N_1 = N_2 + 5g = 68.6\,\text{N} \qquad (4)$$

Adding Equation 1 and Equation 3 , we obtain



$$40 - F_k = 7a \quad \Rightarrow \quad 40 - m_k N_1 = 7a \quad \Rightarrow a = \frac{40 - 0.2 \times 68.6}{7} = 3.75\text{m}/\text{s}^2$$

Substituting for a in Equation 1, we get

$$f_s = 2a = 7.5\,N \quad \Rightarrow \quad m_s = \frac{f_s}{N_2} = \frac{7.5}{19.6} = \ 0.38$$

CHAPTER 5
CIRCULAR MOTION and GRAVITATION

5.1 CENTRIPETAL FORCE

It is known that if a particle moves with constant speed *v* in a circular path of radius *r*, it acquires a centripetal acceleration due to the change in the direction of the particle's velocity.
The direction of this acceleration is toward the center of the circle and given by.

Along the radius

$$a_r = \frac{v^2}{r}$$

Centripetal force

From Newton's 2nd Law this acceleration should be due to force called the centripetal Force.

$$F_r = ma_r = m\frac{v^2}{r} \quad \Rightarrow$$

It should be noted that any force in nature can be treated as a centripetal force if it acts on a particle in a direction toward the center of the circular path followed by the particle.

Remark The centripetal referring to the direction of the force and not to a new kind of forces. It is like horizontal or vertical.

Example 5.1 A flat (unbanked) curve on a highway has a radius of 100 m. If the coefficient of static-friction between the tires and the road is 0.2, what is the maximum speed the car will have in order to round the curve successfully?

Solution Here there are three forces acting on the car:

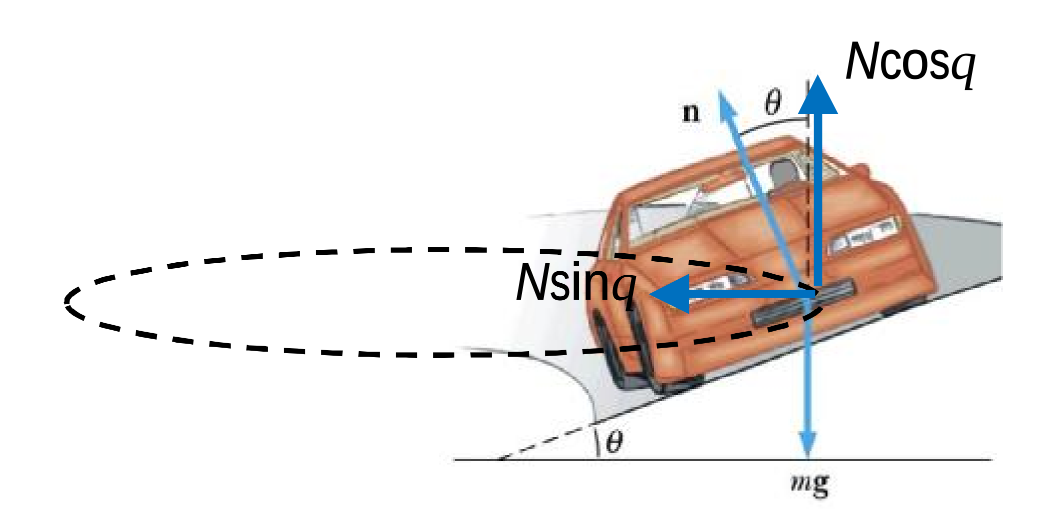
The weight and the normal force act perpendicular to the plane of motion, and the static frictional force parallel to the road.

The static frictional force is the only force that acts along the radius of the curve $\Rightarrow$

$$f_s = m\frac{v^2}{r} \quad (f_s = m_s N) \qquad \Rightarrow \qquad m_s N = m\frac{v^2}{r}$$

since there is no motion in the vertical direction $\Rightarrow$

$$N = mg \qquad \Rightarrow m_s(mg) = m\frac{v^2}{r} \qquad \Rightarrow \qquad v = \sqrt{m_s g r} = 14\,\mathrm{m/s} = 50.4\,\mathrm{km/h}$$



Example 5.2 A circular curve of a road is designed for traffic moving at 60 km/hr without depending on the friction. If the radius of the curve is 80 m, what is the correct angle of road's banking?

Solution Note that the car will move around a horizontal circle ⇒

The normal force *N* should be resolved into two components: one toward the center of the curve (horizontal), and the other vertical.

The centripetal force then is the horizontal component ⇒

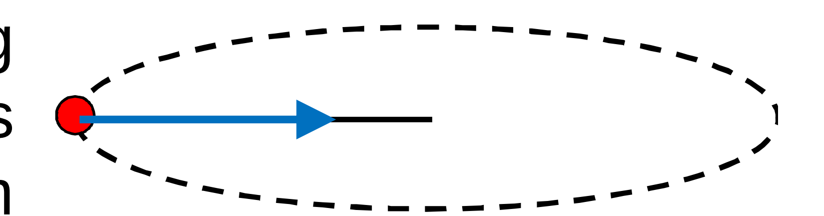
$$N \sin q = m\frac{v^2}{r} \qquad \Rightarrow$$

since there is no motion in the vertical direction $\Rightarrow$

$$N\cos q = mg \quad \Rightarrow \quad N = \frac{mg}{\cos q} \quad \Rightarrow$$

$$mg\tan q = m\frac{v^2}{r} \quad \Rightarrow \quad q = \tan^{-1}\frac{v^2}{gR} = 19.5^{o}$$

Example 5.3 A ball of mass 1 kg is attached to one end of a string 1 m long and is whirled in a horizontal circle, as shown in Figure 5.3. Find the maximum speed the ball can attain without breaking the string. The breaking strength of the string is 500 N.



Solution The only two forces acting on the ball are the weight and the tension.

Since the weight is normal to the plane of the circle, the centripetal force in this case is the tension, so we can write

$$T = m\frac{v^2}{r} \quad \Rightarrow$$

To find the speed at the verge of breaking, we have to substitute for *T* by its breaking value, i.e.,

$$v_{\max} = \sqrt{\frac{T_{\max} R}{m}} = \sqrt{\frac{500 \times 1}{1}} = 22.4 \text{ m/s}$$

5.2 NONUNIFORM CIRCULAR MOTION

When the magnitude of the velocity is not constant but change with time we have the nonuniform circular motion.

The change in the speed will add another contribution to the acceleration.

Resolving the acceleration vector into two perpendicular components: radial component and tangential component ⇒

$$\boldsymbol{a} = a_r \hat{\mathbf{r}} + a_q \hat{q}$$

Unit vector along the radius

Unit vector along the tangent

The radial component, , is the centripetal acceleration defined previously, and the tangential component, , is the new contribution due to the change in the magnitude of the particle's velocity, so we will expect

$$a_q = \frac{d|\mathbf{v}|}{dt}$$

Remark In applying Newton's second law for the circular motion, the coordinate axes will be the radius-axis and the tangent-axis, so all the applied forces have to be resolved accordingly. The law now reads

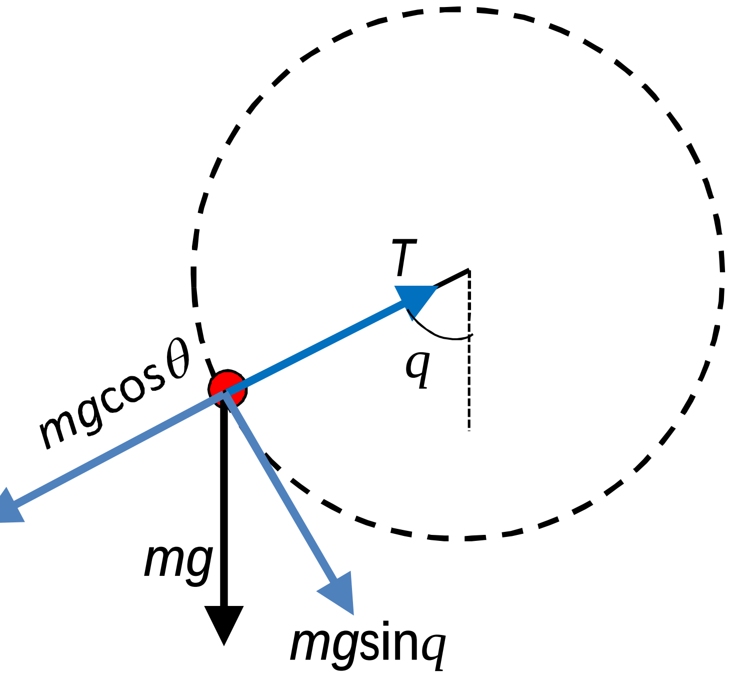
$$\sum F_r = ma_r \qquad \sum F_q = ma_q$$

The positive senses of **r** and q will be chosen toward the center, and counterclockwise respectively.

Example 5.4 A small body of mass *m* swings in a vertical circle at the end of a cord of length *L* as shown. If the speed of the body when the cord makes an angle with the vertical is *v*, find

a) the radial and the tangential components of the acceleration at this point,

b) the tension in the cord at the same point.



Solution The free body diagram of the block is shown.

mg has to be resolved into a radial and a tangential component

For the radial component of the acceleration we have

$$a_r = \frac{v^2}{r} = \frac{v^2}{L}$$

To find the tangential component of the acceleration we use

$$\sum F_\theta = ma_\theta \qquad \Rightarrow mg\sin\theta = ma_\theta \qquad \Rightarrow$$

$$a_\theta = g\sin\theta$$

b) To find the tension we use

$$\sum F_r = ma_r \qquad \Rightarrow$$

$$T - mg\cos\theta = m\frac{v^2}{L} \qquad \Rightarrow T = m\left(\frac{v^2}{L} + g\cos\theta\right)$$

Note that at the bottom point θ=0 the tension is maximum with

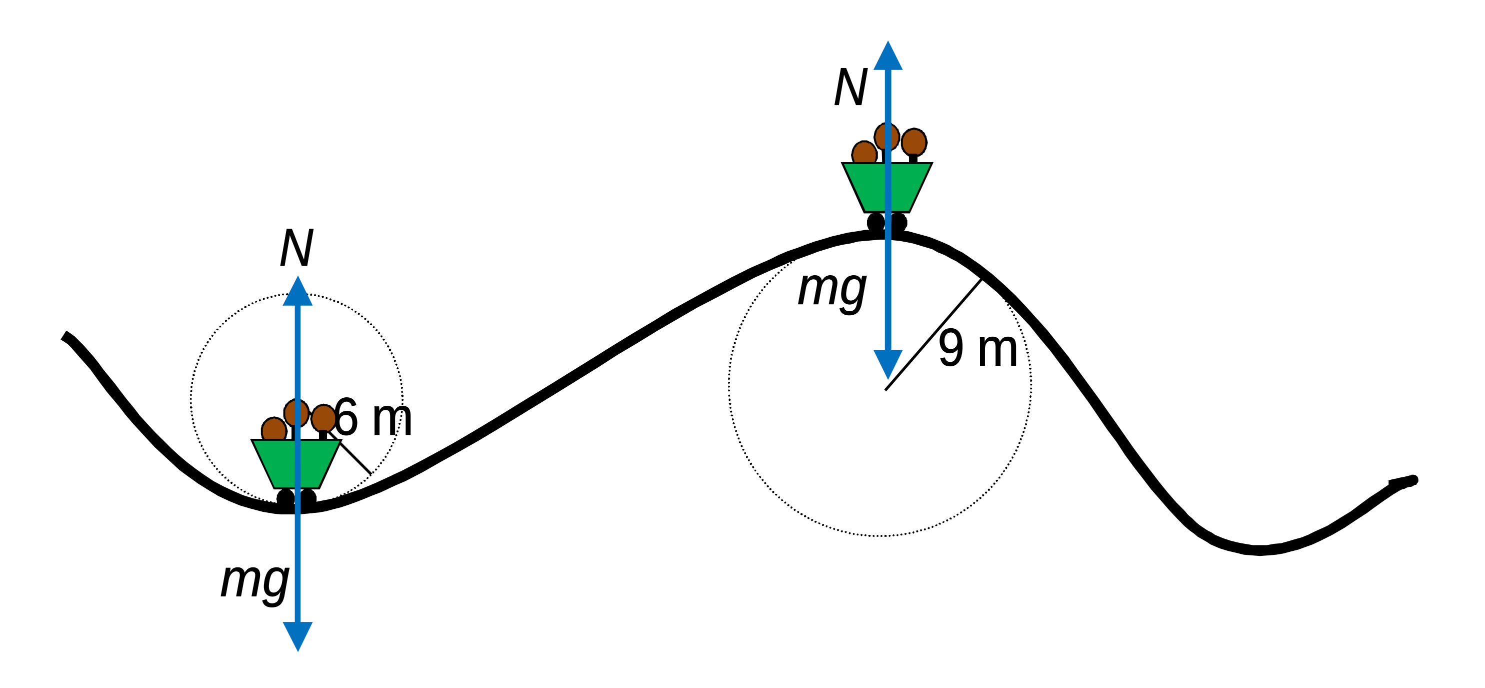
$$T = m\left(\frac{v^2}{L} + g\right)$$

While at the top point θ=π the tension is minimum with

$$T = m\left(\frac{v^2}{L} - g\right)$$

What happen if at the top point $v^2 \leq Lg$





Example 5.5 vehicle of mass 350 kg moves on a roller-coaster as shown in Figure 5.5.

a) If the speed of the vehicle at point *A* is 18 m/s, what is the normal force the track exerts on the vehicle?

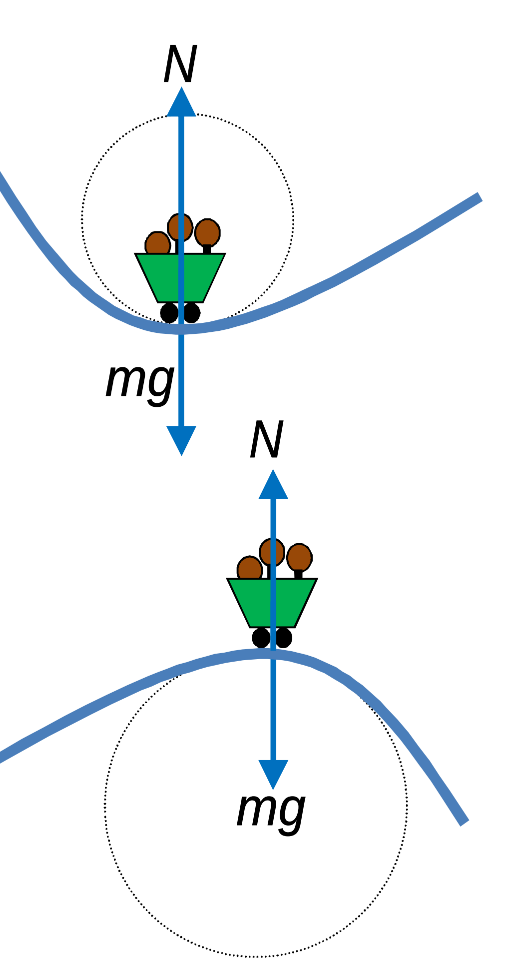
b) What is the maximum speed for the vehicle to remain on track at point *B*?

Solution The free body diagram of the vehicle is shown at the two positions.

a) At point A , *N* is toward the center , while *mg* is away from the center. Applying the eq.

$$\sum F_r = ma_r \quad \Rightarrow \quad N - mg = m\frac{v^2}{r} \quad \Rightarrow$$

$$N = m\left(\frac{v^2}{r} + g\right) = 2.23\times10^3\ \text{N}$$



b) At point B *mg* is toward the center , while *N* is away from the center. Applying again the eq.

$$\sum F_r = ma_r \quad \Rightarrow \quad mg - N = m\frac{v^2}{r}$$

For the vehicle to be on track, the normal force must have a positive value, that is, *N* > 0

$$N = m\left(g - \frac{v^2}{r}\right) > 0 \quad \Rightarrow \quad v < \sqrt{gr} = 9.39\,\text{m}$$

5.3 NEWTON'S LAW OF GRAVITATION

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Thus the gravitational force exerted on a particle of mass m_1 by a particle of mass m_2 is

$$\boldsymbol{F}_{12} = G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}$$

$\hat{\mathbf{r}}$ is a unit vector directed from m_1 to m_2 and G is called the gravitational constant with a value $G = 6.672\times10^{-11}$ N.m^2/kg^2.

The force exerted by any homogeneous sphere is the same as if the entire mass of the sphere is concentrated at its center.

Therefore, the force exerted by the earth on a small body of mass m, a distance r from its center, is

$$F = G\frac{M_e m}{r^2} \qquad r \geq R_e$$

where M_e and R_e are the earth's mass and the earth's radius, respectively.

At the center of the earth the gravitational force on the body would be zero, why?

From Newton's second law, and assuming a body of mass m at the surface of the earth, we have

$$\sum F = G\frac{M_e m}{R_e^2} = mg \quad \Rightarrow \quad g = G\frac{M_e}{R_e^2} \quad \Rightarrow \quad M_e = \frac{R_e^2 g}{G} = 5.96\times10^{24}\ \text{kg}$$

If the body is at a distance h above the earth's surface is then

$$\sum F = G\frac{M_e m}{(R_e + h)^2} = mg' \quad \Rightarrow \quad g' = G\frac{M_e}{(R_e + h)^2}$$

Therefore, g' decrease with increasing altitude.

Example 5.6 Two bodies of mass 60 kg, and 80 kg are placed 2 m apart. Calculate the gravitational force exerted by one body on the other.

Solution Using $F = G\dfrac{m_1 m_2}{r^2} \quad \Rightarrow$

$$F = \left(6.67\times10^{-11}\right)\frac{(60)(80)}{(2)^2} = 8\times10^{-8}\ \text{N}$$

Example 5.7 Three bodies of mass 2 kg, 4 kg, and 6 kg are arranged as shown. Calculate the total force acting on the 2-kg mass by the other two masses.

Solution m_1 is acted upon by two forces:

$$\boldsymbol{F}_{12} = G\frac{m_1 m_2}{r_{12}^2}\,\mathbf{i} = \left(6.67\times10^{-11}\right)\frac{2\times4}{(2)^2}\,\mathbf{i} = 1.33\times10^{-10}\,\mathbf{i}\ \text{N}$$

$$\boldsymbol{F}_{13} = G\frac{m_1 m_3}{r_{13}^2}\,\mathbf{j} = \left(6.67\times10^{-11}\right)\frac{2\times6}{(1)^2}\,\mathbf{j} = 8.0\times10^{-10}\,\mathbf{j}\ \text{N}$$

The total force acting on m_1 is:

$$\boldsymbol{F}_1 = \boldsymbol{F}_{12} + \boldsymbol{F}_{13} = (1.33\mathbf{i} + 8.0\mathbf{j}) \times 10^{-10}\,\mathrm{N}$$

Example 5.8 Calculate the magnitude of the acceleration due to gravity at an altitude of 100 km.

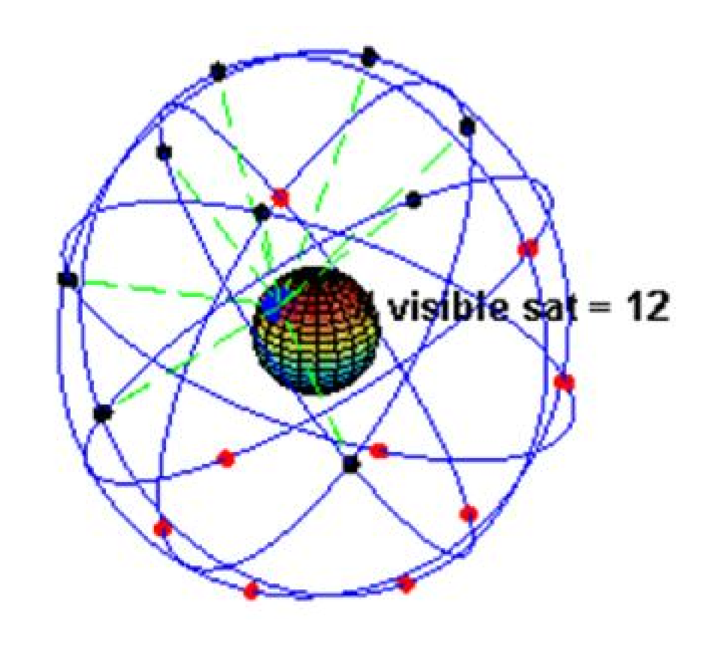
Solution Using the equation

$$g' = G\frac{M_\mathrm{e}}{(R_e + h)^2} \quad \Rightarrow$$

$$g' = \frac{(6.67\times10^{-11})(5.98\times10^{24})}{(6.37\times10^{6} + 1\times10^{5})^2} = 9.5\,\mathrm{m/s^2}$$

5.4 SATELLITE MOTION

Satellite is an object orbiting around the earth.



In satellite motion the gravitational force between the satellite and the earth is the centripetal force. Now applying $\Rightarrow$

$$\sum F_r = m\frac{v^2}{r} \qquad \Rightarrow \qquad G\frac{mM_e}{r^2} = m\frac{v^2}{r} \qquad \Rightarrow$$

where *m* is the mass of the satellite and *r* is the radius of the satellite orbit. Solving for *v* we get

$$v = \sqrt{\frac{GM_e}{r}}$$

The period of revolution is $t = \frac{2p\,r}{v} \qquad \Rightarrow \quad t = 2p\sqrt{\frac{r^3}{GM_e}} = \frac{2p\,r^{3/2}}{\sqrt{GM_e}}$

It should be clear that the previous considerations are also applicable to the motion of our moon around the earth and the motion of the planets around the sun.

Example 5.9 If one want to place a communication satellite into a circular orbit of radius 6800 km. What must be its speed, and its period?

Solution Using the equation

$$v = \sqrt{\frac{GM_e}{r}} \quad \Rightarrow \quad v = \sqrt{\frac{(6.67\times10^{-11})(5.98\times10^{24})}{6.8\times10^{6}}} = 7.66\times10^{3}\ \text{m/s}$$

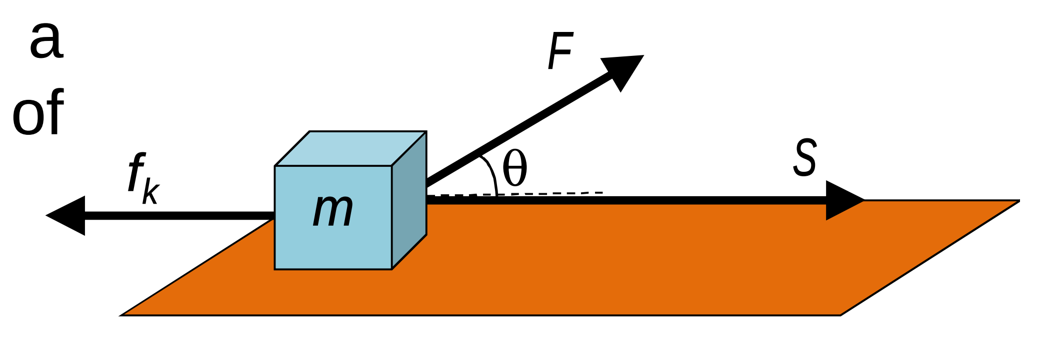
And for the period we use

$$t = \frac{2pr^{3/2}}{\sqrt{GM_\text{e}}} \quad \Rightarrow t = \frac{2p\left(6.8\times10^{6}\right)^{3/2}}{\sqrt{6.67\times10^{-11}\times5.98\times10^{24}}} = 1.55\,\text{h}$$

CHAPTER 6
WORK AND ENERGY

6.1 WORK

Consider an object displaced a distance ***S*** under the action of the constant force ***F*** as shown



The work done by this force is defined as

$$W = \boldsymbol{F} \cdot \boldsymbol{S} = FS\cos q$$

⇒The work is a scalar quantity with its SI unit of N.m or Joule (J).

The work done by a force is zero when the force is perpendicular to the displacement

The work may be positive, or negative depending on the direction of ***F*** relative to ***S***. It is positive if is in the direction of ***S***, and is negative if is in the opposite direction of ***S***.

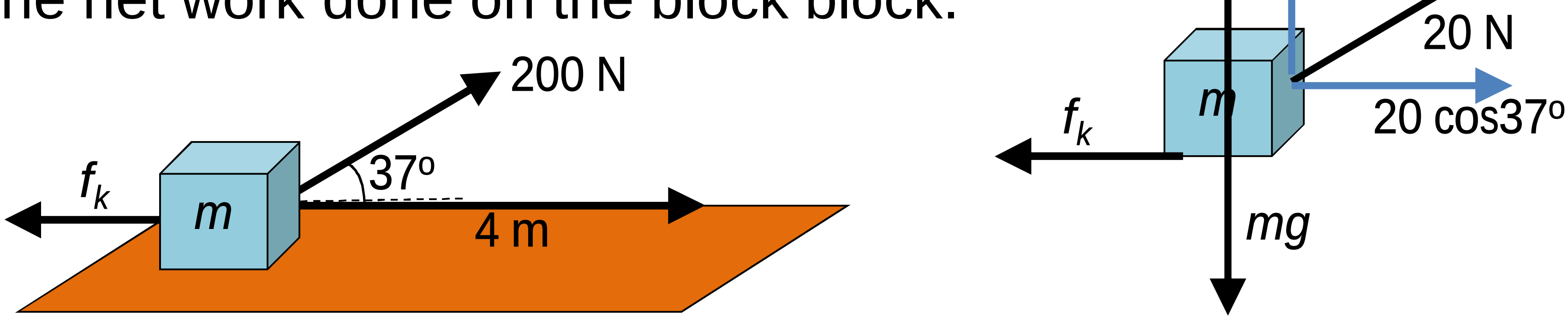
An example of the negative work is the work done by a frictional force.

Example 6.1 A block of mass 2 kg moves under the influence of a force F= 20 N, which makes an angle of 37 above the horizontal. The block moved a distance of 4 m on a rough surface of m_k =0.2. Calculate,

a) the work done by F

b) the work done by friction.

c) the net work done on the block block.



Solution Let us first draw a free body diagram for the block.c

Only the horizontal component of F do work given by

$$W_F = (F \cos q)S = (20 \cos 37^o)4 = 63.9\,\text{J}$$

b) To find the magnitude of the frictional force we have

$$f_k = m_k N$$

but $N + 20\sin 37^o = mg \quad \Rightarrow$

but $N = mg - 20\sin 37^o = 7.6\,\text{N} \quad \Rightarrow$

$$f_k = (0.2)(7.6)) = 1.52\,\text{N}$$

Now $W_f = -f_k S = -(1.52)4 = -6.08\,\text{J} \quad \Rightarrow$

c) Since *mg* and *N* don’t do any work $\Rightarrow$

$$W_{net} = W_f + W_F = 63.9 - 6.08 = 57.8\,\text{J}$$

6.2 WORK DONE BY A VARYING FORCE

If the force is not constant over the displacement we have to divide the displacement into small elements each of *dx*. The work during one of these elements is

$$dW = F_x dx$$

Now the total work done during the total displacement is

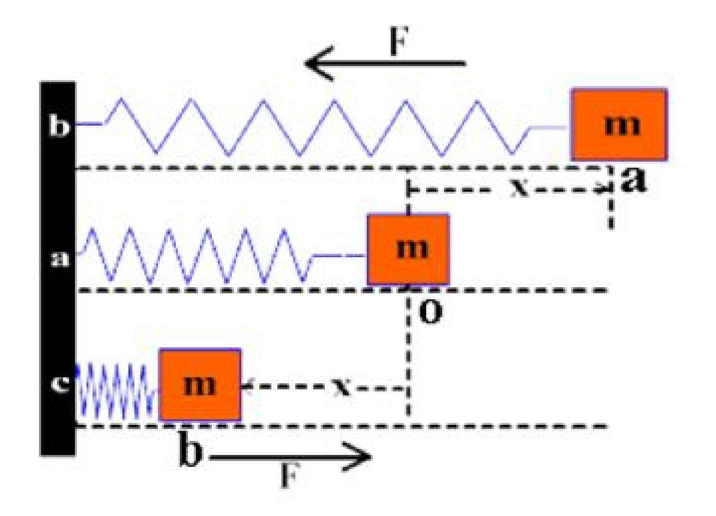
$$W = \int dW = \int_i^f F_x dx$$

In general we write

$$W = \int_i^f \boldsymbol{F} \cdot d\boldsymbol{S}$$

Note that if ***F*** is constant we recover the first eq.

6.3 WORK DONE BY A SPRING



When a block attached to a spring the spring affect the block with a force given by Hook's law:

$$F_s = -kx$$

where x is the displacement of the body from its equilibrium position (x=0) and the force constant k is a measure of the stiffness of the spring.

The minus sign in Hook's law tells that the force of the spring is always opposes the displacement.

Let us calculate the work done by the force F_s , as the body moves from an initial position x_i to a final position x_f

$$W_s = \int_i^f F_s dx = -k \int_{x_i}^{x_f} x dx \quad \Rightarrow W_s = \frac{1}{2} k \left(x_i^2 - x_f^2 \right)$$

From the last eq. we can conclude that

(i) If the mass moves toward the equilibrium position then $x_f = 0$ and the work is positive. This is due to the fact that the force and the displacement, in this case, are in the same direction.

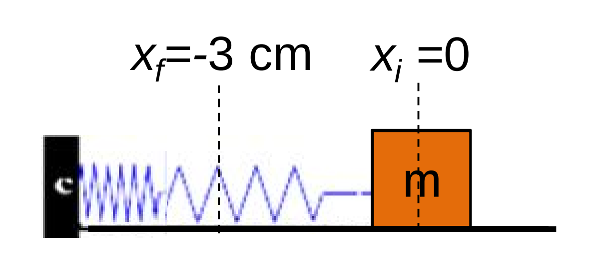
(ii) If the mass moves a way from the equilibrium position then $x_i = 0$, and the work is negative. Here the force and the displacement are in opposite directions.

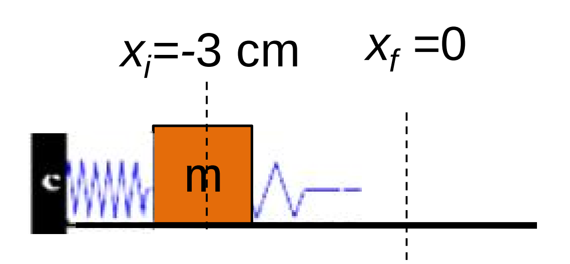
Remark: The work done by an external force in compressing or stretching a spring is equal to the negative of the work done by the spring's force during the corresponding displacement.

Example 6.2 A block is tied to a spring with force constant of 80 N/m as shown. The spring is compressed a distance 3 cm from equilibrium position.

a) Calculate the work done by the spring as the block moves from its equilibrium to its compressed position.

b) Calculate the work done by the spring as the block returns to its equilibrium position.





Solution a) The block was at its equilibrium position (x_i =0) and moves to a final position (x_f =-3 cm) ⇒

$$W_s = \frac{1}{2}k\left(x_i^2 - x_f^2\right) \quad = \frac{80}{2}\left(0 - 9\times10^{-4}\right) = -3.6\times10^{-2}\,\text{J}$$

b) Now (x_i =-3cm) and moves to a final position (x_f =0) ⇒

$$W_s = \frac{1}{2}k\left(x_i^2 - x_f^2\right) \quad = \frac{80}{2}\left(9\times10^{-4} - 0\right) = 3.6\times10^{-2}\,\text{J}$$

6.4 WORK-KINETIC ENERGY THEOREM

Let us consider that the net force acting on an object is in *x* direction, then we write

$$W_{net} = \int_i^f F_x dx$$ Using Newton's second law we have

$$F_x = ma_x = m\frac{dv}{dt} \quad \text{but } \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \Rightarrow F_x = mv\frac{dv}{dx} \quad \Rightarrow$$

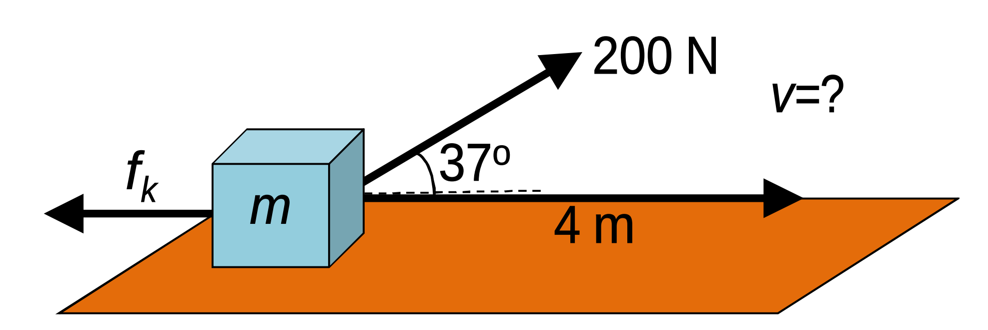
$$W_{net} = m\int_i^f v\frac{dv}{dx}dx = m\int_i^f vdv = \frac{1}{2}mv^2\Big|_{v_i}^{v_f} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad \Rightarrow$$

$$\boxed{W_{net} = K_f - K_i = \Delta K}$$

with $K = \frac{1}{2}mv^2$ Is called the kinetic energy

Example 6.3 A block of mass 2 kg moves under the influence of a force F= 20 N, which makes an angle of 37 above the horizontal. If the initial velocity is zero, find the final velocity after the block moves a distance 4 m.

Solution In example 6.1 we found that



$$W_{net} = 57.8\,\mathrm{J}$$

Applying the work-energy theorem, we have

$$W_{net} = K_\mathrm{f} - K_\mathrm{i} = \tfrac{1}{2}m\left(v_f^2 - v_i^2\right) \quad \Rightarrow$$

zero

$$57.8 = \tfrac{1}{2}(2)v_f^2 \quad \Rightarrow$$

$$v_f = \sqrt{57.8} = 7.6\,\mathrm{m/s^2}$$

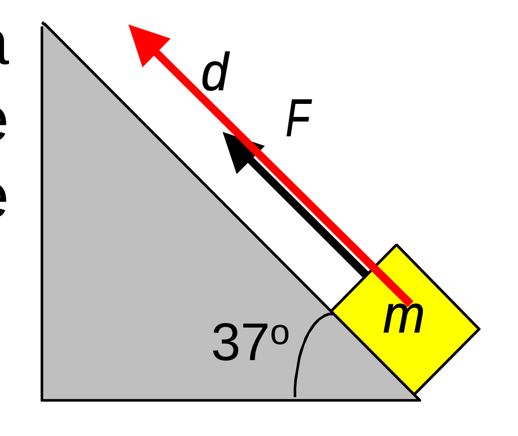
Example 6.4 A mass of 2 kg is pushed up a rough inclined plane by a force F=20 N. The mass is displaced a distance 0.5 m on the inclined plane. Calculate,

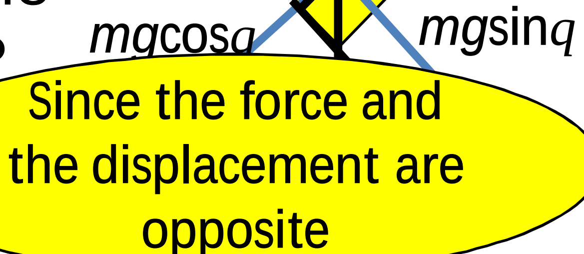
a) the work done by the force of gravity,

b) the work done by the force F=20 N,

c) the work done by friction if m_k=0.2,

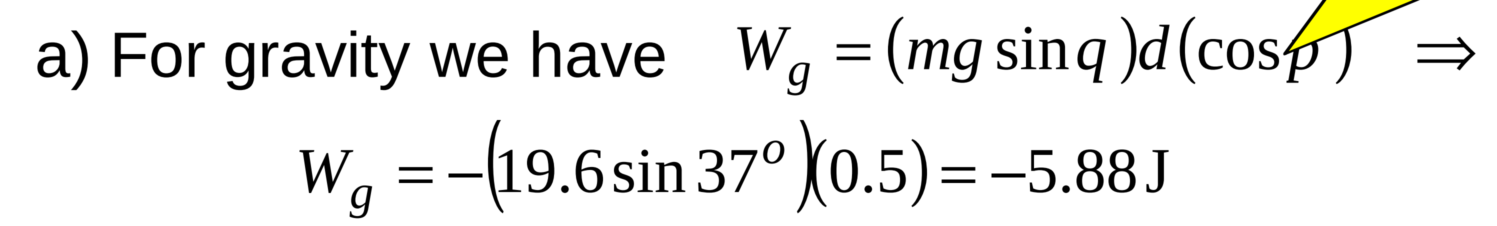
d) If the mass has a kinetic energy of 1.2 J at the beginning of the displacement, what is the kinetic energy at the end of the displacement?





Solution Let us first draw the fee-body di[agram for] the block.

Since the force and the displacement are opposite



a) For gravity we have $W_g = (mg\sin q)d(\cos p) \Rightarrow$

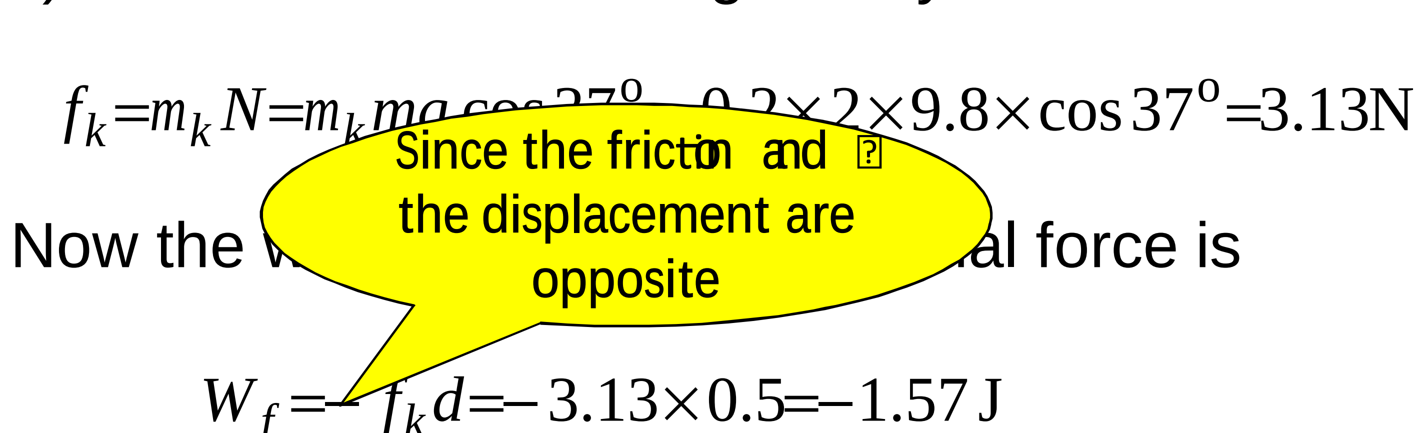
$$W_g = -(19.6\sin 37^o)(0.5) = -5.88\,\text{J}$$

Note that the second component of gravity doesn't do any work since its perpendicular to the displacement.

b) **F** is in the same direction as the displacement ***S***, so the work done by ***F*** is,

$$W_F = Fd = 20\times 0.5 = 10\,\text{J}$$

c) The force of friction is given by



$$f_k = \mu_k N = \mu_k mg\cos 37^{o} = 0.2\times 2\times 9.8\times\cos 37^{o} = 3.13\text{N}$$

Now the w… al force is

$$W_f = -f_k d = -3.13\times 0.5 = -1.57\,\text{J}$$

d) Using the work-kinetic energy theorem we have

$$\Delta K = W_{\text{net}} = W_{\text{g}} + W_{\text{F}} + W_f = -5.88 + 10 - 1.57 = 2.55\,\text{J}$$

$$\text{but}\ \Delta K = K_f - K_i \qquad \Rightarrow$$

$$K_f = \Delta K + K_i = 2.55 + 1.2 = 3.75\,\text{J} \qquad \Rightarrow$$

6.5 POWER

The power is defined as time rate at which work is done.

The average power during the time interval is defined as

$$\overline{P} = \frac{\Delta W}{\Delta t}$$

The instantaneous power is defined as

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

If ***F*** is constant, then dW = ***F***.d***S*** $\Rightarrow P = \frac{dW}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{S}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v}$

The unit of power is J/s or watt, (W), with 1W=1 J/s=1 kg.m^2/s^3 .

Example 6.5 A 1500-kg car accelerates uniformly from rest to a speed of 10 m/s in 3 s. Find

a) the work done on the car in this time,

b) the average power delivered by the engine in the first 3 s,

c) the instantaneous power delivered by the engine at t=3 s.

Solution a) Using the work-kinetic energy theorem we have

zero

$$W_{\text{net}} = \Delta K = \tfrac{1}{2} m\left(v_f^2 - v_i^2\right) = \tfrac{1}{2}(1500)(10)^2 = 7.5\times10^4\ \text{J}$$

b) Now we have $\overline{P} = \dfrac{\Delta W}{\Delta t} = \dfrac{7.5\times10^4}{3} = 2.5\times10^4\ \text{W}$

c) For the instantaneous power we have $P = \boldsymbol{F}\cdot\boldsymbol{v} = m\boldsymbol{a}\cdot\boldsymbol{v}$

Now using $v = v_o + at \Rightarrow a = \dfrac{v - v_o}{t} = \dfrac{10-0}{3} = 3.33\,\text{m/s}^2$

$$\Rightarrow P = m\boldsymbol{a}\cdot\boldsymbol{v} = (1500)(3.33)(10) = 5\times10^4\ \text{W}$$

CHAPTER 7
CONSERVATION OF ENERGY

7.1 CONSERVATIVE AND NON-CONSERVATIVE FORCES

Conservative Force: The force is conservative if the work done by it on a particle that moves between two points depends only on these points and not on the path followed.

As an example of a conservative force is the force of the spring.

The work done by such a force is

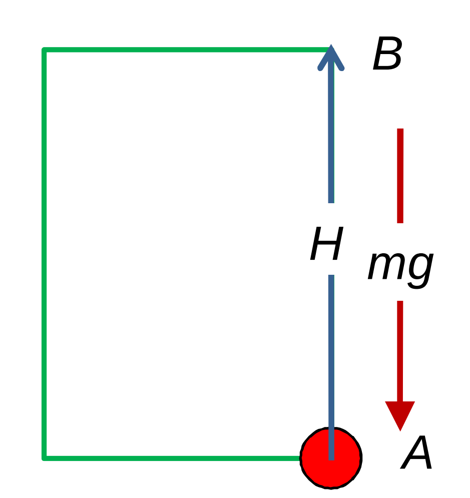
$$W_s = \frac{1}{2}k\left(x_i^2 - x_f^2\right)$$

As it is clear from the Equation, the work done by a spring depends only on x_i and x_f .

As another example of a conservative force is the force of gravity.

Consider a particle goes from point *A* to a point *B* .

The work done by gravity if the particle followed path 1, (the blue path) is simply $W_1 = -mgH$



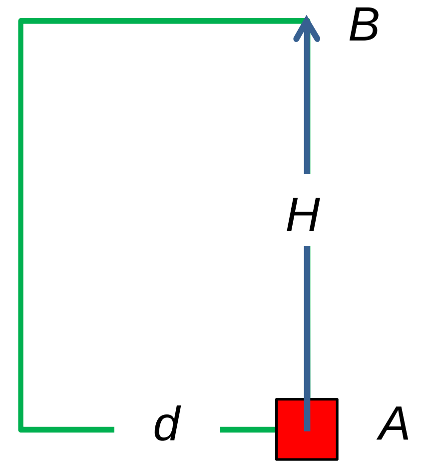
If the particle followed another path, the green path, we see again that the work done by gravity is also $W_2 = -mgH$

Non-conservative Force: The force is non-conservative if the work done by that force on a particle that moves between two points depends on the path taken between those points.

The Frictional force is an example of a non-conservative force.

Since the frictional force is always tangent to the path and opposite to the motion its work depends on the path followed.

If a block is to move from A to B along a rough horizontal surface following two paths then



$$W_1 = -f_k H$$

$$\text{while } W_2 = -f_k H - 2 f_k d$$

7.2 POTENTIAL ENERGY

For every conservative force we associate a potential energy such

$$W_C = -\Delta U = -(U_f - U_i)$$

With *U* is called the potential energy.

7.3 GRAVITATIONAL POTENTIAL ENERGY

Since the force of gravity is conservative, we can associate a gravitational potential energy function U_g to this force. To calculate such potential energy, we consider a particle that makes a displacement from an initial point *i* to a final point *f*,

The force of gravity does a negative work given by *-mgh* (why negative?).

$$-mgh = -(U_f - U_i) \quad \Rightarrow \quad mgh = U_f - U_i$$

f

h

mg

i

Since we are concerned only by the change in the potential energy, U_i can be chosen to be zero.

The horizontal level at which U_i is zero is called the level of zero potential energy. $\Rightarrow$

$$U_g = mgh$$

where h is the vertical displacement above an arbitrary horizontal level (level of zero potential energy).

Remark: The gravitational potential energy is positive if the body is above the level of zero potential energy, and negative if the body is below the level.

Example 7.1 A book of mass 1.2 kg is on a horizontal table that is 1.5 m high. The book and the table are in a room of height 2.8 m.
a) What is the gravitational potential energy of the book if the level of zero potential energy is taken to be (i) at the table's surface, (ii) at the floor, and (iii) at the ceiling?
b) When the book drops to the floor, calculate for the three choices of the zero potential energy.

Solution a) The block is at the table $\Rightarrow$

(i) If the table is the level of zero P.E. $\Rightarrow$ $U_g = mgh = 0$

(ii) If the floor is the level of zero P.E. $\Rightarrow$

$$U_g = mgh = 1.2 \times 9.8 \times 1.5 = 17.64\,\mathrm{J}$$

(iii) If the ceiling is the level of zero P.E. $\Rightarrow$

$$U_g = -mgh = -1.2 \times 9.8 \times 1.3 = -15.29\,\mathrm{J}$$

a) Now the block moved from the table to the floor $\Rightarrow$

(i) If the table is the level of zero P.E. $\Rightarrow$

$$U_i = 0 \; and \; U_f = -mgh = -17.64\,\mathrm{J} \qquad \Rightarrow \Delta \mathrm{U} = -17.64\,\mathrm{J}$$

(ii) If the floor is the level of zero P.E. $\Rightarrow$

$$U_i = 1.2 \times 9.8 \times 1.5 = 17.67\,\mathrm{J} \text{ and } U_f = 0 \quad \Rightarrow \Delta U = -17.64\,\mathrm{J}$$

(iii) If the ceiling is the level of zero P.E. $\Rightarrow$

$$U_i = -1.2 \times 9.8 \times 1.3 = -15.29\,\mathrm{J}$$

$$\text{and} \qquad U_f = -1.2 \times 9.8 \times 2.8 = -32.93\,\mathrm{J} \quad \Rightarrow$$

$$\Delta U = -17.64\,\mathrm{J}$$

Note that DU is independent of the level.

7.4 POTENTIAL ENERGY OF A SPRING

Consider a particle attached to a spring and goes from the equilibrium position (x_i = 0) to a position where the spring is compressed or stretched a distance x (x_f = x) , the force of spring does a work given by

$$W_s = \tfrac{1}{2}k\left(x_i^2 - x_f^2\right) = -\tfrac{1}{2}kx^2$$

$$U_s = -W_s = \tfrac{1}{2}kx^2$$

With U_i is taken to be zero at the equilibrium position.

Remarks: (i) The potential energy of the spring is zero at the equilibrium point (x = 0).

(ii) U_s is always positive, since x^2 is always positive.

7.5 CONSERVATION OF MECHANICAL ENNERGY

Suppose that a particle moves under the influence of a conservative force *F*. $\Rightarrow$

$$W_{net} = W_C = \Delta K$$

But since the force is conservative $\Rightarrow$

$$W_C = -\Delta U = \Delta K \quad \Rightarrow$$

$$\boxed{\Delta K + \Delta U = 0}$$

or $\left(K_f - K_i\right) + \left(U_f - U_i\right) = 0 \quad \Rightarrow \quad \left(K_f + U_f\right) - \left(U_i + K_i\right) = 0 \quad \Rightarrow$

$$\boxed{E_f - E_i = 0}$$

with $E = K + U$ Is the mechanical energy

Example 7.2 A small block of mass m = 2 kg is released from a height of h = 10 m above the ground as shown. Using the law of conservation energy determine,

a) the speed of the block at an altitude of y = 4 m above the ground,

b) the velocity of the ball just before it hits the ground.

Solution a) Applying the conservation of energy principle between points A and B, $\Rightarrow$

$$(K_A + U_A) = (U_B + K_B) \Rightarrow \quad (0 + mgh) = \left(mgy + \tfrac{1}{2} m v_B^2\right) \Rightarrow$$

$$v_B = \sqrt{2g(h-y)} = \sqrt{19.6(10-4)} \quad = 10.8\ \text{m/s}$$

b) Now applying the conservation of energy principle between points A and C, $\Rightarrow$

$$(K_A + U_A) = (U_C + K_C) \quad \Rightarrow \quad (0 + mgh) = \left(0 + \tfrac{1}{2} m v_C^2\right) \quad \Rightarrow$$

$$v_C = \sqrt{2gh} = \sqrt{19.6(10)} \quad = 14\ \text{m/s}$$

7.6 NONCONSERVATVE FORCE AND WORK-ENERGY THEOREM

If there are conservative and non-conservative forces acting on a system, and these forces do work. $\Rightarrow$

$$W_{net} = W_C + W_{NC}$$

$$\text{but}\, W_{net} = \Delta K \;\; and \;\; W_C = -\Delta U \quad \Rightarrow \quad \Delta K = -\Delta U + W_{NC} \quad \Rightarrow$$

$$\boxed{\Delta K + \Delta U = W_{NC}} \quad \boxed{\text{or} \quad \Delta E = W_{NC}}$$

Strategy for solving problems using Newton's laws:

(i) Select a horizontal zero level for gravitational potential energy.

(ii) Define two points: an initial point and a final point.

(iii) Find the *U* and *K* at these two points.

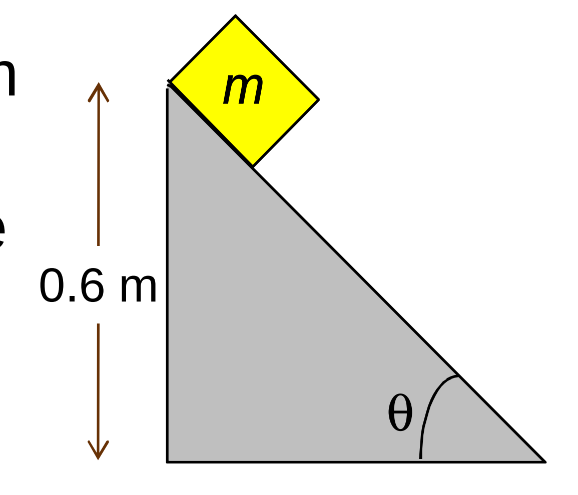
(iv) If there is a spring, then $U = U_g + U_s$

(v) If there are friction, then calculate W_{nc}. If not then $W_{nc} = 0$.

Example 7.3 A 2-kg mass slides down a rough inclined plane, as shown. the mass starts from rest, and the friction force is given by $f = 5$ N.

a) Use energy method to find the speed of the block at the bottom of the incline.

b) If the inclined plane is frictionless find the speed at that point.



Solution a) The ground is chosen as the level for the zero potential energy, and the initial point is chosen at the top of the plane, while the final point is chosen at the bottom of the plane. ⇒

$$E_{\mathrm{i}} = K_{\mathrm{i}} + U_{\mathrm{i}} = \tfrac{1}{2} m v_i^2 + m g y_i \quad = 0 + 2 \times 9.8 \times 0.6 = 11.76\,\mathrm{J}$$

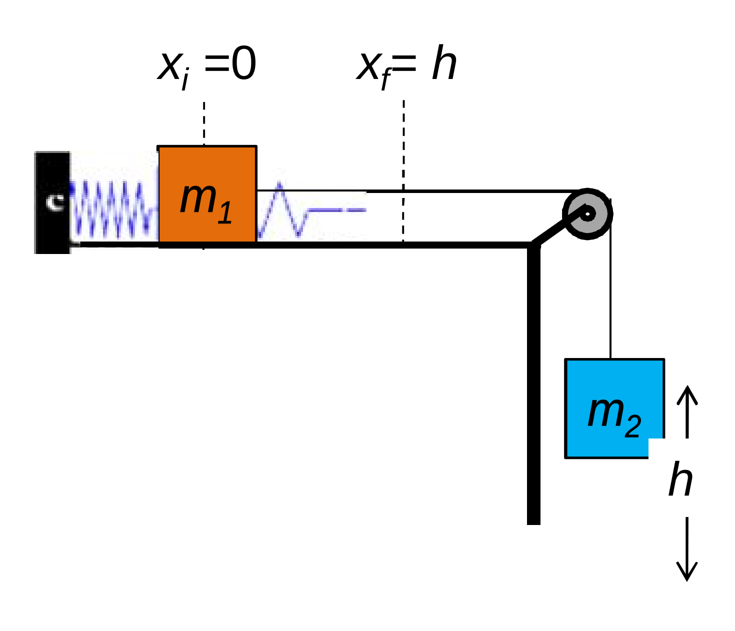
$$E_{\mathrm{f}} = K_{\mathrm{f}} + U_{\mathrm{f}} = \tfrac{1}{2} m v_f^2 + 0 = v_f^2 \qquad W_{\mathrm{nc}} = -fs = -5 \times \frac{0.6}{\sin 37} = -4.98\,\mathrm{J}$$

$$\text{Now} \quad \Delta E = W_{NC} \Rightarrow \; v_f^2 - 11.76 = -4.98 \Rightarrow$$

b) If there is no friction ⇒ $W_{\mathrm{nc}} = 0$

$$v_f^2 - 11.76 = 0 \Rightarrow \qquad v_f = \sqrt{11.76} = 3.43\,\mathrm{m/s}$$

Example 7.3 Two blocks are connected by a light string that passes over a frictionless pulley as shown. The mass m_1 lies on a rough surface, and the system is released from rest when the spring is unstretched (x = 0). The mass m_2 falls a distance h before coming to rest. Calculate the coefficient of kinetic friction between m_1 and the surface.



Solution since the system starts from rest and ends at rest $\Rightarrow$

$$\Delta K = 0 \qquad \text{Now } \Delta U = \Delta U_g + \Delta U_s \quad \Rightarrow$$

$$\Delta U_1 = \Delta U_s = \tfrac{1}{2}kh^2 \text{ and } \Delta U_2 = \Delta U_g = -m_2 gh \Rightarrow \quad \Delta U = \tfrac{1}{2}kh^2 - m_2 gh$$

The work done by friction is $\quad W_{NC} = -fh = -(mm_1 g)h$

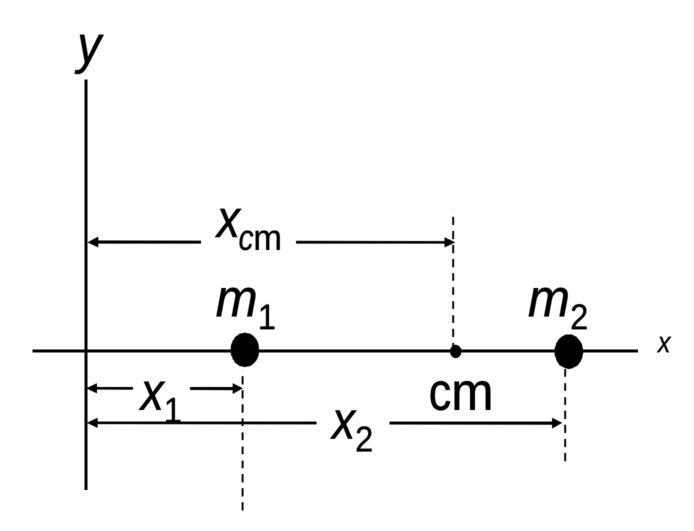
$$\text{Now} \quad \Delta E = W_{NC} \Rightarrow \tfrac{1}{2}kh^2 - m_2 gh = -mm_1 gh \quad \Rightarrow \quad m = \frac{m_2 g - \frac{1}{2}kh}{m_1 g}$$

CHAPTER 8
SYSTEMS OF PARTICLES

8.1 CENTER OF MASS

The center of mass of a system of particles or a rigid body is the point at which all of the mass are considered to be concentrated there and all external forces were applied there.

Consider a system of two masses m_1 and m_2 located along the x-axis as shown. The position x_{cm} of the center of mass of these two masses is defined to be



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

For a system of n-particles m_1, m_2, …, m_n, the center of mass is

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots m_n x_n}{m_1 + m_2 + \ldots m_n} \qquad x_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

Where $M = m_1 + m_2 + \ldots + m_n$ Is the total mass of the system

If the system is distributed on three dimension, the *y* and the *z* coordinates of the center of mass are similarly defined by

$$y_{cm} = \frac{1}{M}\sum_{\mathrm{i}=1}^{\mathrm{n}} m_{\mathrm{i}} y_{\mathrm{i}} \qquad z_{cm} = \frac{1}{M}\sum_{\mathrm{i}=1}^{\mathrm{n}} m_{\mathrm{i}} z_{\mathrm{i}}$$

In vector notation, the position vector of the center of mass $\boldsymbol{r}_{\mathrm{cm}}$ can be expressed as

$$\mathbf{r}_{\mathrm{cm}} = x_{\mathrm{cm}}\mathbf{i} + y_{\mathrm{cm}}\mathbf{j} + z_{\mathrm{cm}}\mathbf{k} \qquad \text{or } \mathbf{r}_{cm} = \frac{1}{M}\sum_{\mathrm{i}=1}^{\mathrm{n}} m_{\mathrm{i}}\mathbf{r}_{\mathrm{i}}$$

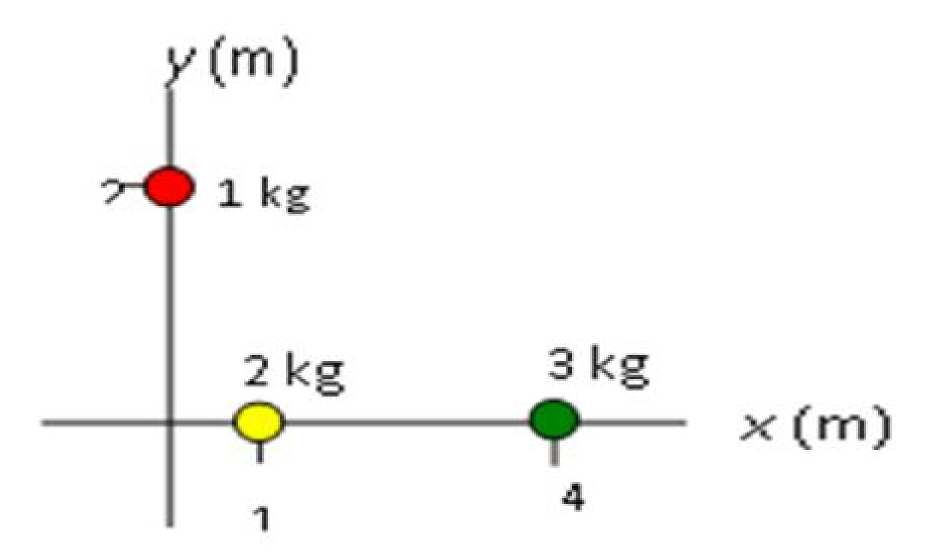
For a rigid body (continuous mass distribution) we treat the body as consisting of so large number of small elements each of mass *dm* such that the sums become integrals⇒

$$x_{cm} = \frac{1}{M}\int x dm \qquad y_{cm} = \frac{1}{M}\int y dm \qquad z_{cm} = \frac{1}{M}\int z dm$$

With *x,y*, and *z* are the coordinates of the element *dm*. For the vector position of the center of mass we write

$$\boldsymbol{r}_{cm} = \frac{1}{M}\int \boldsymbol{r} dm$$

Example 8.1 Three particles of masses $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg are located as shown in Figure 8.10. Find the center of mass of this system.



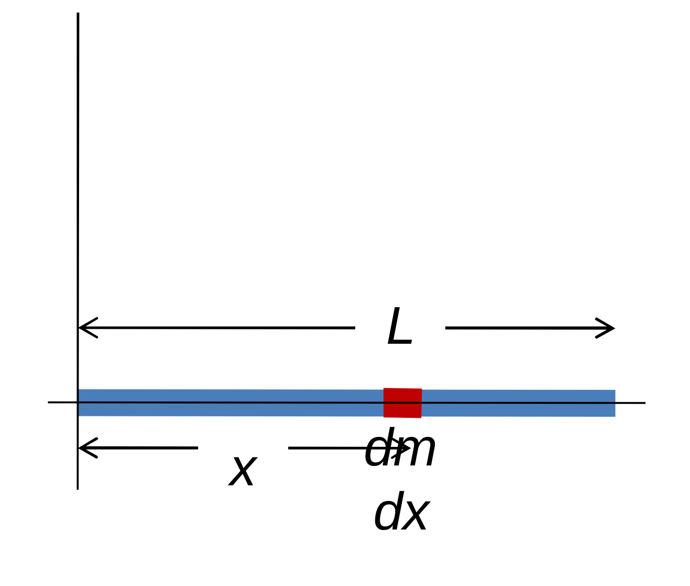
Solution The *x*-component of the center of mass is

$$x_{\rm cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1\times 0 + 2\times 1 + 3\times 4}{1+2+3} = \frac{14}{6} = 2.33\,\text{m}$$

$$y_{\rm cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{1\times 2 + 2\times 0 + 3\times 0}{1+2+3} = \frac{2}{6} = 0.33\,\text{m}$$

$$\mathbf{r}_{\rm cm} = (2.33\mathbf{i} + 0.33\mathbf{j})\,\text{m}$$

Example 8.2 Show that the center of mass of a uniform rod of mass *M* and length *L* lies midway between its ends.



Solution Let the rod be located along the *x*-axis such that $y_{cm} = z_{cm} = 0$.

Let us take a small element of mass *dm* and length *dx*.

$$\text{Now } x_{cm} = \frac{1}{M}\int x\,dm$$

To find a relation between the mass *dm* and the variable *x* we define the linear mass density λ (mass per unit length), as

$$l = \frac{dm}{dx} = \frac{M}{L} \quad \Rightarrow \quad dm = l\,dx$$

$$x_{\text{cm}} = \frac{l}{M}\int_0^L x\,dx \quad \Rightarrow \quad x_{\text{cm}} = \frac{l}{M}\frac{x^2}{2}\Bigg]_0^L = \frac{l\,L^2}{2M} = \left(\frac{M}{L}\right)\frac{L^2}{2M} = \frac{L}{2}$$

8.2 DYNAMICS OF A SYSTEM OF PARTICLES

It is known that $$\mathbf{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i \quad \Rightarrow \quad M\mathbf{r}_{cm} = \sum_{i=1}^{n} m_i \mathbf{r}_i$$

Differentiating the last equation with respect to time gives

$$M\boldsymbol{v}_{cm} = \sum_{i=1}^{n} m_i \boldsymbol{v}_i$$

Differentiating again with respect to time gives

$$M\boldsymbol{a}_{cm} = \sum_{i=1}^{n} m_i \boldsymbol{a}_i = \sum_{i=1}^{n} \boldsymbol{F}_i$$

with $\boldsymbol{F}_i$ is the vector sum of all the forces acting on the i^{th} particle. From Newton's 3rd law the internal forces form action-reaction pairs so that they cancel out. So, the sum of $\boldsymbol{F}_i$ is the vector sum of all the external forces $\mathbf{F}_{ext}$ that act on the system.

$$M\boldsymbol{a}_{cm} = \sum \boldsymbol{F}_{ext}$$

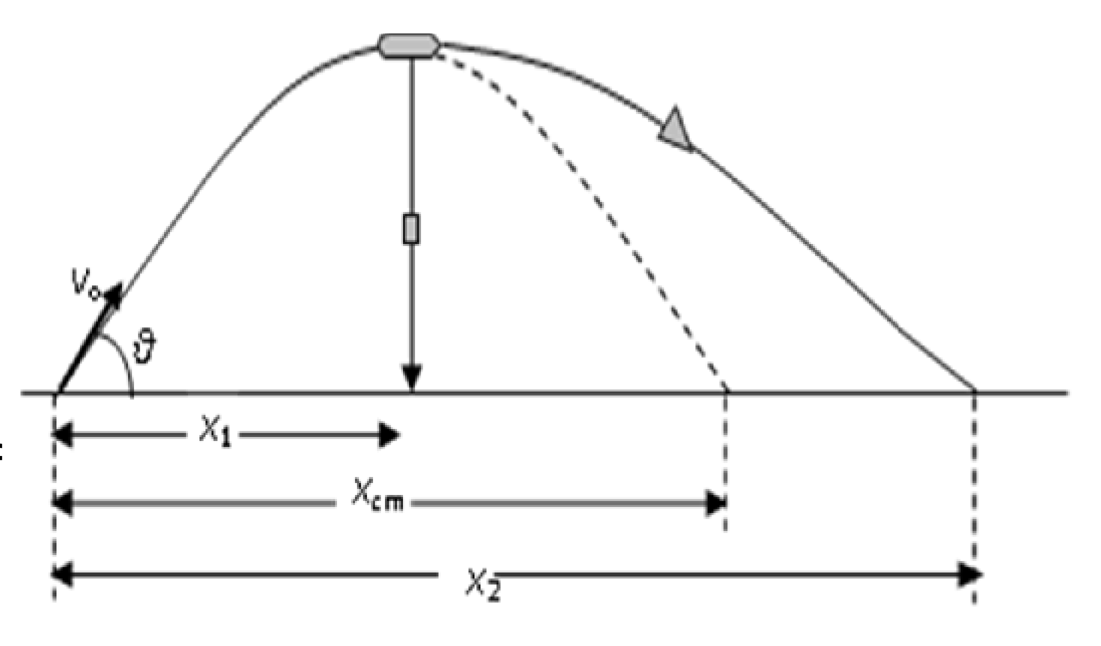
Like any vector equation, the last equation can written as three equations corresponding to the components of $\boldsymbol{a}_{cm}$ and $\boldsymbol{F}_{ext}$ along the coordinates axes, that is,

$$\sum F_x^{ext} = Ma_x^{cm} \qquad \sum F_y^{ext} = Ma_y^{cm} \qquad \sum F_z^{ext} = Ma_z^{cm}$$

The last equations tell that if no net external force acting on a system, the acceleration of its center of mass is zero and thus the velocity of the center of mass of the system remains unchanged.

Example 8.3A shell is fired with an initial speed v_0 at an angle of θ above the horizontal. At the top of the trajectory, the shell explodes into two equal fragments. One fragment, whose speed immediately after explosion is zero, falls vertically down, as shown. How far from the initial point does the other fragment land.

Solution As the forces due to the explosion is internal, they do not affect the motion of the center of mass. Since the only external force acting on the system is the force of gravity, the center of mass follows a parabolic path (the dotted path) as the projectile did not explode. From Example 3.2 we obtain



$$x_{cm} = \frac{v_o^2 \sin 2q}{g}$$

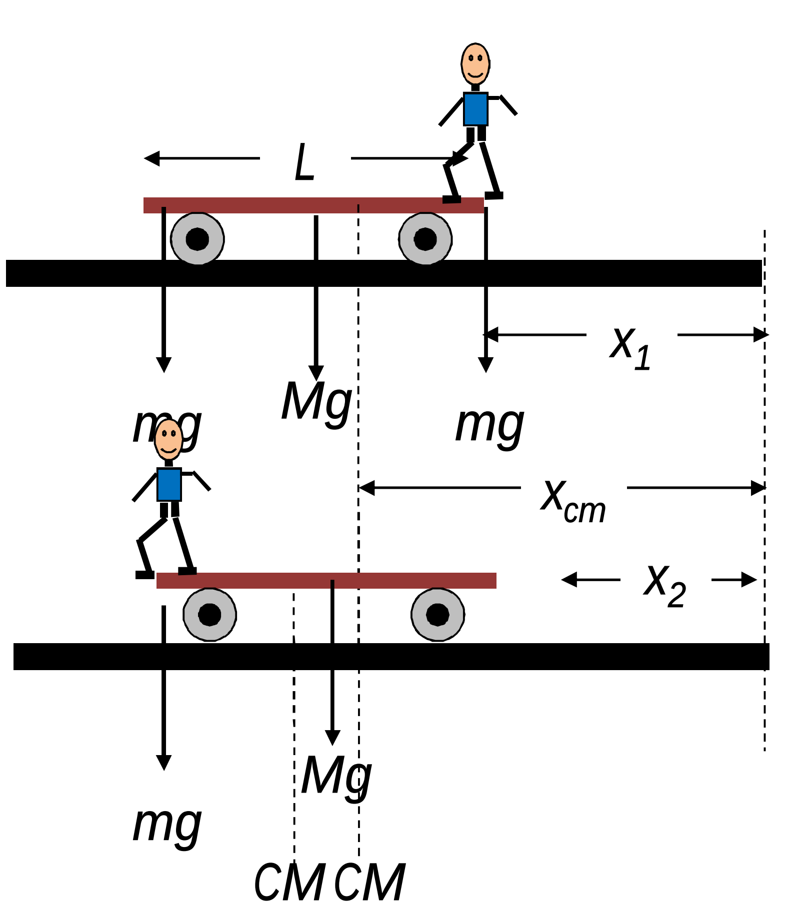
$$\text{But } x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{M}$$

Knowing that $x_1 = \frac{1}{2} x_{cm} \; and \; m_1 = m_2 = \frac{1}{2} M \quad \Rightarrow$

$$x_{\text{cm}} = \frac{\frac{1}{2} x_{cm} + x_2}{2} \quad \Rightarrow \quad x_2 = \frac{3}{2} x_{cm} = \frac{3}{2} \frac{v_o^2 \sin 2q}{g}$$

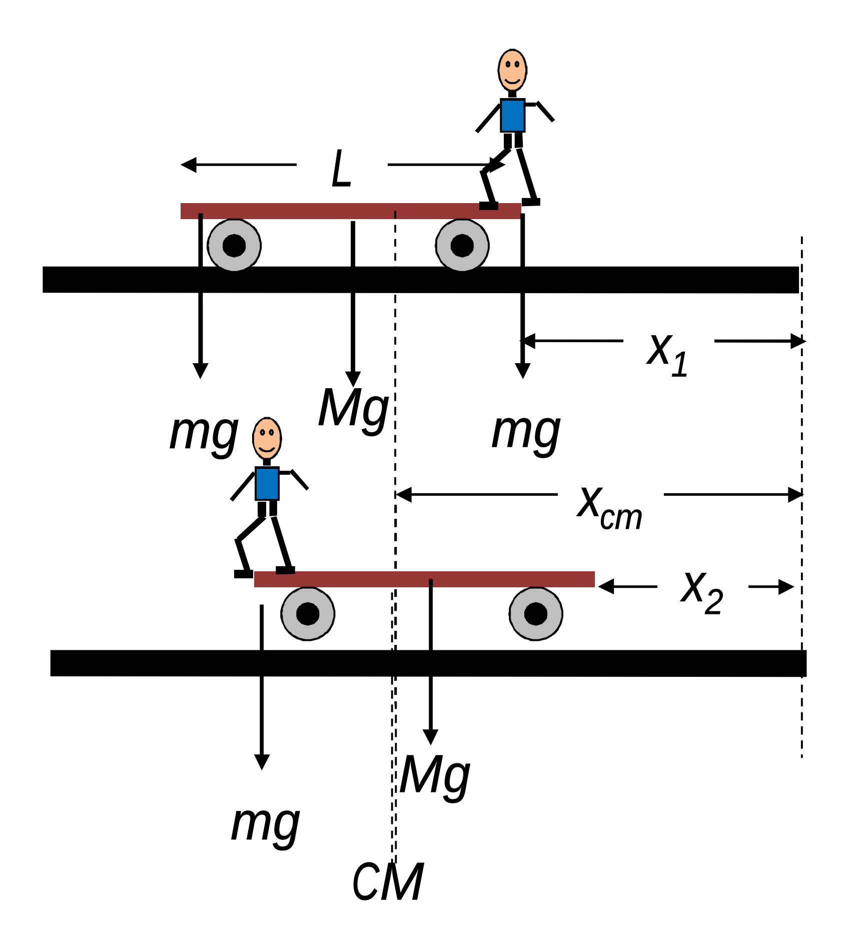
Example 8.4 A car of mass M moves along a smooth horizontal track. A man of mass m is initially standing at one end of the car, which is initially at rest, as shown. If the man starts to walk toward the other end of the car, describe the motion of the car.

Solution There is no external force acting on the man-car system along the horizontal direction. This means that the velocity of the center of mass of the system will not change and must remain zero, and so the position of the center of mass of the man-car system will not change.



Let x_1 be the position of right end of the car relative to a fixed axis, and x_2 is the new position of the same end. the position of the center of mass of the man-car system when the man is at the right end is

$$x_{cm} = \frac{mx_1 + M\left(x_1 + \frac{1}{2}L\right)}{m + M}$$



When the man is now at the left end of the car the position of the center of mass is

$$x_{cm} = \frac{m(x_2 + L) + M\left(x_2 + \frac{1}{2}L\right)}{m + M}$$

Equating the above two equations we obtain

$$x_2 = x_1 - \frac{m}{M + m}L$$

The last equation tells that if the man moves a distance L to the

left, the car will move to the right a distance $\frac{m}{M + m}L$

8.3 LINEAR MOMENTUM

The linear momentum ***p*** of a particle of mass *m* moving with velocity ***v*** is defined as

$$\boldsymbol{p} = m\boldsymbol{v}$$

In SI unit system the momentum has a unit of kg.m/s.

From Newton's second law we have

$$\sum \boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d(m\boldsymbol{v})}{dt} \quad \Rightarrow \quad \sum \boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

For a system of n particles, each with its own mass and velocity, the linear momentum ***P*** of the system as a whole is the vector sum of the linear momenta of each particle individually, that is

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots + \boldsymbol{p}_n \quad = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + \cdots + m_n\boldsymbol{v}_n = \sum_{\mathrm{i}=1}^{\mathrm{n}} m_\mathrm{i}\boldsymbol{v}_\mathrm{i}$$

$$\text{But } M\boldsymbol{v}_{cm} = \sum_{\mathrm{i}=1}^{\mathrm{n}} m_\mathrm{i}\boldsymbol{v}_\mathrm{i} \quad \Rightarrow \quad \boldsymbol{P} = M\boldsymbol{v}_{cm}$$

Differentiating the last equation with respect to time we get

$$\frac{d\boldsymbol{P}}{dt} = M\frac{d\boldsymbol{v}_{cm}}{dt} = M\boldsymbol{a}_{cm} \quad (\sum \boldsymbol{F}_{ext}) \quad \Rightarrow \quad \sum \boldsymbol{F}_{ext} = \frac{d\boldsymbol{P}}{dt}$$

8.4 CONSRVATION OF LINEAR MOMENTUM

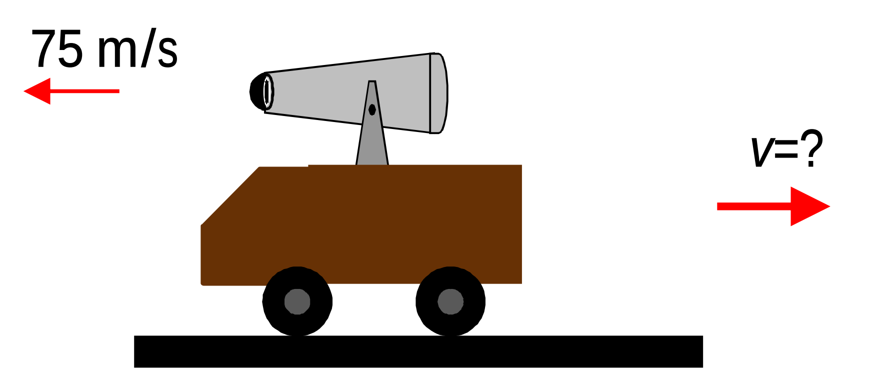
If a system is isolated, that is the resultant external force acting on the system is zero ⇒

$$\sum \boldsymbol{F}_{ext} = \frac{d\boldsymbol{P}}{dt} = 0 \quad \Rightarrow \quad \boldsymbol{P} \text{ is constant} \Rightarrow \boldsymbol{P}_i = \boldsymbol{P}_f$$

For an isolated system, the linear momentum of the system at an initial point *i* is equal to the linear momentum at a final point *f*.

If only one component of the net external force acting on a system along an axis is zero, then the component of the linear momentum of the system along that axis is constant only. The other two components in this case is not constant.

Example 8.5 A cannon of mass 2000 kg rests on a smooth, horizontal surface. The cannon fires, horizontally, a ball of mass 25 kg with a speed of 75 m/s relative to the earth. What is the velocity of the cannon just after it fires the ball?



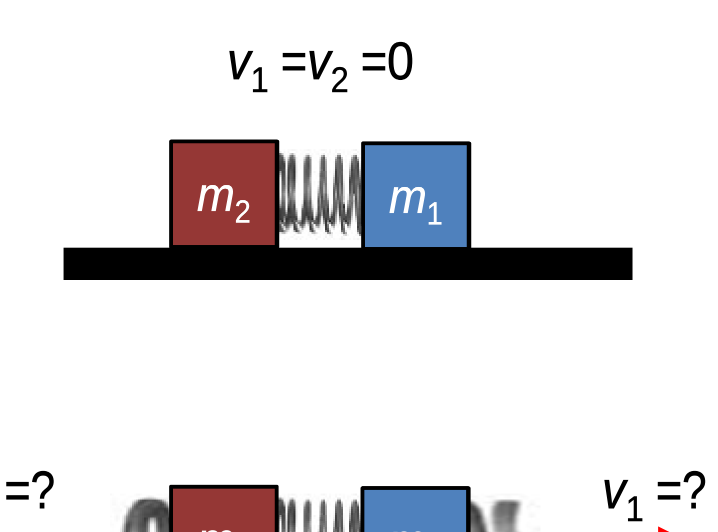
Solution We take our system to consists of the cannon and the cannonball. There are no horizontal external forces acting on the system. The two external forces, the force of gravity and the normal force, are both vertical. Therefore, the *x*-component of the linear momentum of the system is conserved $\Rightarrow P_i = P_f$

$$\text{But } \boldsymbol{P}_i = 0 \quad \text{and } \boldsymbol{P}_f = m_c v_c + m_b v_b$$

With *c* and *b* refer, respectively, to the cannon and the cannonball.

$$\Rightarrow m_c v_c + m_b v_b = 0 \qquad \Rightarrow v_c = -\frac{m_b}{m_c} v_b = -\left(\frac{25}{2000}\right) 75 = -0.94 \text{ m/s}$$

Example 8.6 Two blocks of masses m_1=1kg and m_2= 2kg are connected by a spring of k=200 N/m. The two blocks are free to slide along a smooth horizontal surface. The blocks are initially compressing the spring 12 cm, and then released from rest. Find the velocities of the two blocks when the spring returns to its equilibrium state.



Solution Our system is the two blocks and the spring. Again no external forces acting on the system on the horizontal direction. The total momentum in the horizontal direction is therefore, conserved $\Rightarrow P_i = P_f$. Knowing that the system is initially at rest we obtain

$$0 = m_1 v_1 + m_2 v_2 \quad \Rightarrow \quad v_2 = -\frac{m_1}{m_2} v_1$$

Now applying the conservation of mechanical energy principle $\Rightarrow$

$$\frac{1}{2}kx^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad \Rightarrow \quad m_1 v_1^2 + m_2\left(\frac{m_1}{m_2}\right)^2 v_1^2 = kx^2$$

$$v_1 = \sqrt{\frac{km_2 x^2}{m_1 m_2 + m_1^2}} = \sqrt{\frac{(200)(2)(0.12)}{2+1}} = 4.0\,\text{m/s}$$

Again using the relation between the two velocities we get

$$v_2 = -\frac{m_1}{m_2}v_1 \quad = -\frac{1}{2}(4.0) = -2.0\,\text{m/s}$$

CHAPTER 9
COLLISIONS

9.1 IMPULSE

When two objects crash into each other each object exerts a force on the other. From Newton's 2nd law we have

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \quad \Rightarrow \; d\boldsymbol{p} = \boldsymbol{F}dt \quad \Rightarrow$$

Integrating the last Eq. $\Rightarrow$

$$\int_{i}^{f} d\boldsymbol{p} = \int_{t_i}^{t_f} \boldsymbol{F}dt \quad \Rightarrow \; \Delta\boldsymbol{p} = \boldsymbol{J}$$

with $\boldsymbol{J} = \int_{t_i}^{t_f} \boldsymbol{F}dt$ Is defined as the impulse of the force.

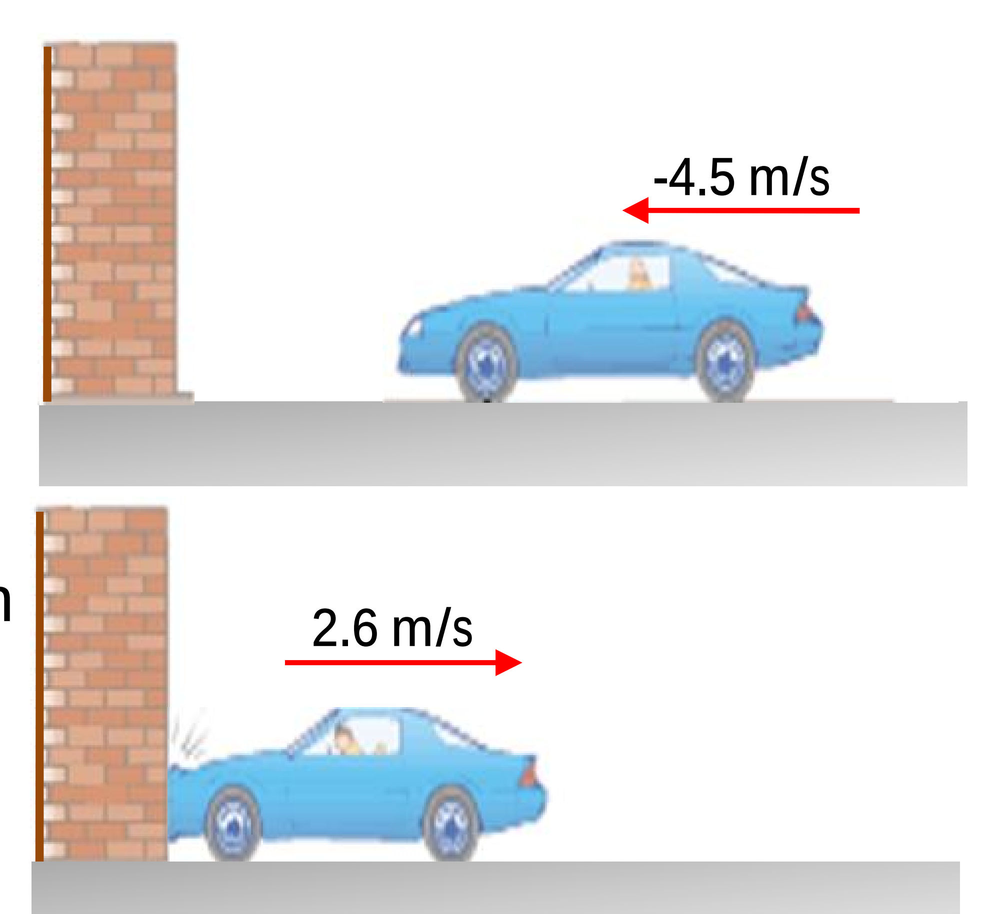
As the force varies with time, we can define the average force as

$$\boldsymbol{J} = \Delta\boldsymbol{p} = \overline{\boldsymbol{F}}\Delta t$$

Example 9.1 A car of mass 1500 kg collides with a wall as shown. The initial velocity is v_i =4.5 m/s to the left, and final velocity v_f =2.6 m/s to the right.

a) Find the impulse due to the collision.

b) If the average force exerted on the car is 1.76×10^5 N, find the contact time Δt.



Solution *a*) Let us take the positive sense to be to the right. The impulse *J* is

$$J = p_f - p_i = mv_f - mv_i \quad = 1500(2.6 + 4.5) = 1.07\times10^4 \text{ kg m/s}$$

b) From Equation (9.5) we have, $\Delta t = \dfrac{J}{\bar{F}} = \dfrac{1.07\times10^4}{1.76\times10^5} = 60.5\,\text{ms}$

Example 9.2 A ball of mass 200 g is dropped from a height h=1.8 m above the floor. After colliding with the floor, the ball rebound vertically to a height h'=1.2 m.
a) Find the impulse on the ball due to the collision.
b) If the collision last for a period of 12 ms, find the average force exerted on the ball.

1.8 m
1.2 m
v_i
v_f

Solution To find the velocities just before and just after collision, we use the conservation of energy method.

$$K_i + U_i = K_f + U_f$$

For the motion of the ball before colliding with the floor, the last equation gives

$$mgh = \frac{1}{2} m v_i^2 \quad \Rightarrow \quad v_i = \sqrt{2gh} = \sqrt{(2)(9.8)(1.8)} = 5.94\,\text{m/s}$$

And for the motion of the ball after colliding with the floor it gives

$$\frac{1}{2} m v_i^2 = mgh' \quad \Rightarrow \quad v_f = \sqrt{2gh'} = \sqrt{(2)(9.8)(1.2)} = 4.95\,\text{m/s}$$

a) Now for the impulse we have

$$J = \Delta P = p_f - p_i = m(v_f - v_i) \quad = 0.2(4.95 + 5.94) = 2.18\,\text{kg.m/s}$$

b) For the average force is

$$\overline{F} = \frac{J}{\Delta t} = \frac{2.18}{0.012} = 181.7\,\text{N}$$

9.2 COLLISIONS

The collision process represents the event of two particles coming together (in contact) for a short time, producing impulse forces on each other.

Considering the collided particles as one system, the forces produced by the collision is internal forces. Therefore, in collision, the total momentum of the system just before the collision equals the total momentum of the system just after the collision. That is

$$m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}} = m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}}$$

According to the conservation of kinetic energy, two types of collision will be considered:

Elastic collision

In elastic the kinetic energy of the system is conserved. That is

$$\frac{1}{2}m_1v_{1\text{i}}^2 + \frac{1}{2}m_2v_{2\text{i}}^2 = \frac{1}{2}m_1v_{1\text{f}}^2 + \frac{1}{2}m_2v_{2\text{f}}^2$$

Inelastic collision

In elastic the kinetic energy of the system is not conserved. The loss in kinetic energy is expressed as $K_{\mathrm{i}} - K_{\mathrm{f}}$

If the two collided particles stick and move together with the same final velocity *vf* after collision, we have the so called **perfectly inelastic collision**.

The conservation of linear momentum is now written as

$$m_1 \boldsymbol{v}_{1\mathrm{i}} + m_2 \boldsymbol{v}_{2\mathrm{i}} = (m_1 + m_2)\boldsymbol{v}_{\mathrm{f}}$$

This collision is still inelastic collision, i.e., the kinetic energy is not conserved.

9.3 ONE-DIMENSIONAL ELASTIC COLLISIONS

If the collision in one-dimensional we have only one component for velocities and the conservation of linear momentum becomes

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Remark: In applying the above equation we have to take into account the signs of the speeds. v is positive if the body moves to the right, and is negative if the body moves to the left.

Let us consider an elastic collision between two particles. One particle of mass m_1 moving with speed v_{1i} and the other particle of mass m_2 moving with speed $v_{2i} \Rightarrow$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad \Rightarrow$$

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i}) \qquad (1)$$

Because the collision is elastic we can write

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \qquad \Rightarrow$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2) \quad \Rightarrow$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad (2)$$

Dividing Equation (2) by equation (1) yield

$$(v_{1i} + v_{1f}) = (v_{2f} + v_{2i}) \qquad \Rightarrow \; v_{1i} - v_{2i} = v_{2f} - v_{1f} \qquad (3)$$

From equation (3) we conclude that the relative velocity of the two particles before collision, $v_{1i} – v_{2i}$, equals the negative of the relative velocity of the two particles after collision, -($v_{1i} – v_{2f}$).

Now solving Equation (1) and Equation (3), for v_{1f} and v_{2f} we get

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \qquad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}$$

Let us consider few special cases concerning the last two equations:

(i) If $m_1 = m_2$ we obtain $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, if the two masses are equal the particles exchange their velocities.

(ii) If m_2 is initially at rest and is very light compared with m_1 we get $v_{1f} \approx v_{1i}$ and $v_{2f} = 2v_{1i}$. That is, when a very heavy particle collides with a light particle, initially at rest, the heavy particle continues its motion unchanged after collision, while the lighter particle moves with a speed equal to twice the initial speed of the heavy particle.

(iii) If m_2 is initially at rest and is very heavy compared with m_1 we get $v_{1f} \approx -v_{1i}$ and $v_{2f} \ll v_{1i}$. That is, when a very light particle collides with a very heavy particle, initially at rest, the velocity of the light particle will be reversed after collision, while the heavy particle will remain approximately at rest.

Example 9.3 A small particle of mass m_1 =2 kg moving to the right with speed v_{1i} = 12 m/s collides elastically with a block of mass m_2 =6 kg moving to the left with a speed of v_{2i} = 8 m/s . Find the velocities of the two particles after collision.

Solution Taking the positive sense are taken to the right

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

$$v_{1f} = \left(\frac{2-6}{2+6}\right) 12 - \left(\frac{12}{2+6}\right) 8 = -6 - 12 = -18\,\text{m/s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

$$v_{2f} = \left(\frac{4}{2+6}\right) 12 - \left(\frac{6-2}{2+6}\right) 8 = 6 - 4 = 2\,\text{m/s}$$

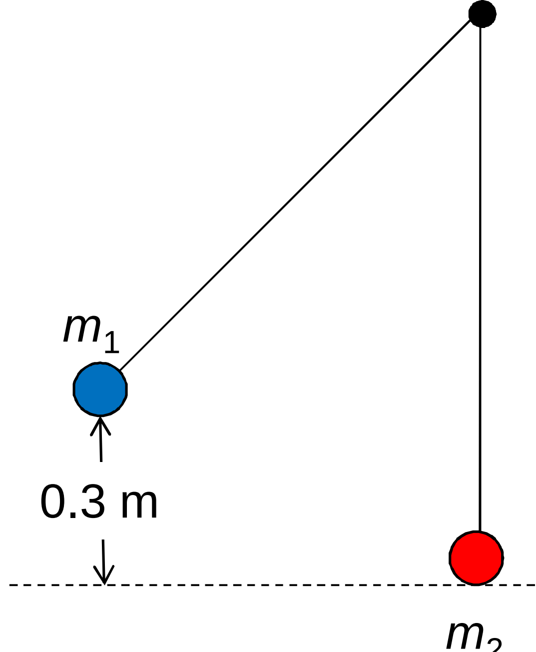
Example 9.4 Two pendulums are initially situated as shown. The pendulum of m_1 =50 kg is released from a height of 30 cm and that of m_2 =2 kg is initially at rest. After swinging down, m_1 undergoes an elastic collision with m_2.



a) Find the velocities of the two balls immediately after collision.



b) To what heights does the two balls swing after collision?



Solution To find the speed of m_1 before collision, v_{1i}, we use the conservation of energy principle:

$$m_1gh = \frac{1}{2}m_1v_{1i}^2 \qquad \Rightarrow \quad v_{1i} = \sqrt{2gh} = \sqrt{19.6 \times 0.3} = 2.42\,\text{m/s}$$

Note that v_{1i} is horizontal despite the motion of m_1 through the two dimensional arc before reaching m_2. Thus, the collision is one dimensional with no horizontal external force acting on the two-pendulums system. To find the velocities of the two balls immediately after collision we apply

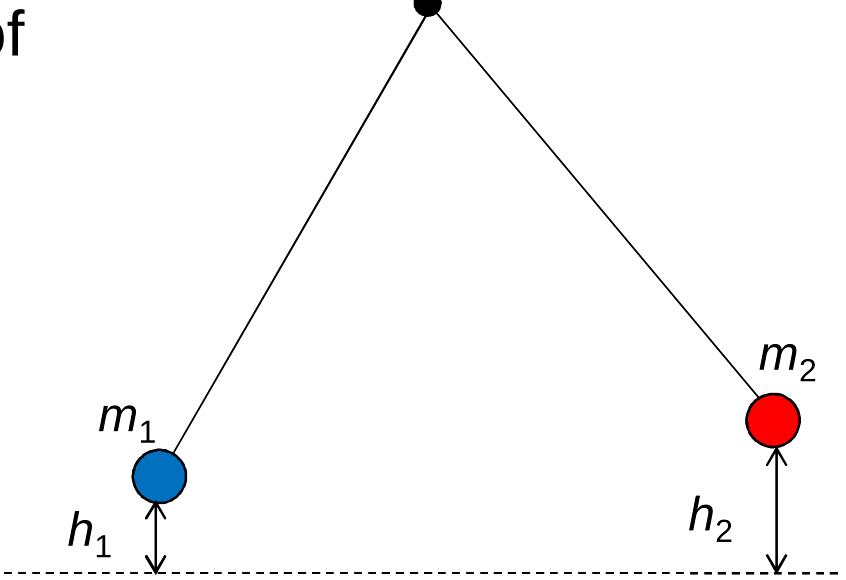
$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i} \qquad v_{1f} = \left(\frac{50-2}{52}\right) 2.42 = 2.23 \, \text{m/s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \qquad v_{2f} = \left(\frac{100}{52}\right) 2.42 = 4.65 \, \text{m/s}$$

Solution Applying the conservation of energy principle for m_1 :

$$\frac{1}{2} m_1 v_{1f}^2 = m_1 g h_1 \quad \Rightarrow$$

$$h_1 = \frac{v_{1f}^2}{2g} = \frac{(2.23)^2}{(2)(9.8)} = 0.25 \, \text{m}$$



Solution And for m_2 :

$$\frac{1}{2} m_2 v_{2f}^2 = m_2 g h_2 \quad \Rightarrow \qquad h_2 = \frac{v_{2f}^2}{2g} = \frac{(4.65)^2}{(2)(9.8)} = 1.1 \, \text{m}$$

9.4 ONE-DIMENSIONAL INEALSTIC COLLISIONS

Consider two particle of masses m_1 and m_2 moving with initial velocities v_{1i} and v_{2i} along a straight line. If the two particles stick together after collision we have the so called perfectly inelastic collision. Their final velocities are then equal, i.e.,

$$v_{1f} = v_{2f} = v_f$$

The conservation of angular momentum is now:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \quad \Rightarrow \quad v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{(m_1 + m_2)}$$

In this type of collision, the kinetic energy of the system is not conserved. That is the total kinetic energy of the system after an inelastic collision is always less than before the collision. The loss of kinetic energy transfers to some other forms of energy, like thermal energy.

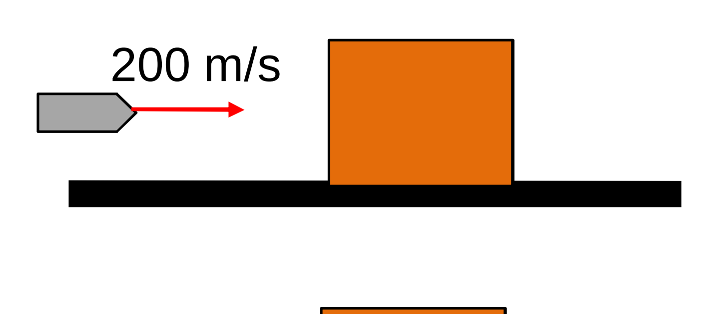
The loss of kinetic energy can be computed as $K_i - K_f$ where

$$K_i = \tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 \qquad \text{and } K_f = \tfrac{1}{2}(m_1 + m_2) v_f^2$$

Example 9.5 A bullet of mass 20 g is fired with a speed of 200 m/s into a block of mass 2 kg being at rest. The bullet is stopped by the block, and the system moves after collision. Find,

a) the velocity after collision, v_f

b) the loss in kinetic energy.



Solution a) Applying the conservation of momentum principle to our system we obtain

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \Rightarrow 0.02 \times 200 + 0 = (2.02) v_f \Rightarrow v_f = 1.98\,\text{m/s}$$

b) to find the loss in kinetic energy, let us calculate K_i and K_f :

$$K_\text{i} = \frac{1}{2} m_1 v_{1\text{i}}^2 + \frac{1}{2} m_2 v_{2\text{i}}^2 = \frac{1}{2} 0.02 \times (200)^2 = 400\,\text{J}$$

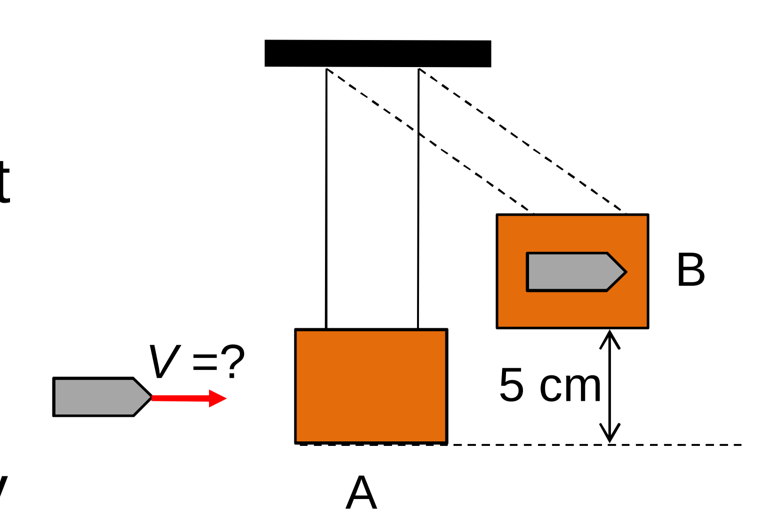
$$K_\text{f} = \tfrac{1}{2}(m_1 + m_2) v_\text{f}^2 = \tfrac{1}{2}(2.02)(1.98)^2 = 3.96\,\text{J}$$

The loss in K.E. is then $K_\text{i} - K_f = 400 - 3.96 = 396\,\text{J}$

Example 9.6 A bullet of m_1 = 5 g is fired toward a block of m_2 = 1 kg, suspended from two light wires, as shown. The bullet is stopped by the block and the system swing through a height h=5 cm.

a) Find the initial speed of the bullet.

b) Calculate the loss in the kinetic energy due to collision.



Solution a) Applying the conservation of linear momentum we get

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \Rightarrow \quad 0.005 v_{1i} + 0 = (1.005) v_f$$

To find v_f we have to apply the conservation of m.energy for the bullet-block system after collision:

$$K_A + U_A = K_B + U_B \Rightarrow \quad \tfrac{1}{2}(m_1 + m_2) v_f^2 + 0 = 0 + (m_1 + m_2) gh \Rightarrow$$

$$v_f = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 0.05} = 0.99 \,\text{m/s} \qquad \Rightarrow v_{1i} = (1.005)\frac{0.99}{0.005} = 199 \,\text{m/s}$$

9.5 TWO-DIMENSIONAL COLLISIONS

If the collided particles move in two dimensions either before, or after collision, the equation of conservation of linear momentum is now resolved into two rectangular components. That is

$$(p_{1\mathrm{i}})_{\mathrm{x}} + (p_{2\mathrm{i}})_{\mathrm{x}} = (p_{1\mathrm{f}})_{\mathrm{x}} + (p_{2\mathrm{f}})_{\mathrm{x}}$$

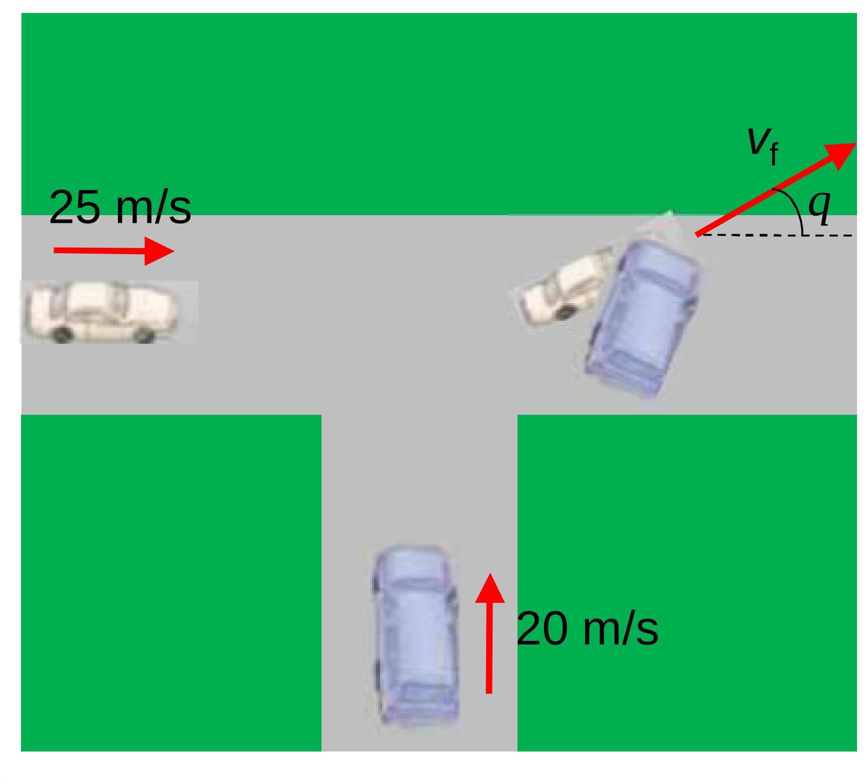
$$(p_{1\mathrm{i}})_{\mathrm{y}} + (p_{2\mathrm{i}})_{\mathrm{y}} = (p_{1\mathrm{f}})_{\mathrm{y}} + (p_{2\mathrm{f}})_{\mathrm{y}}$$

For elastic collision the equation of conservation of kinetic energy is not changed for two dimensional collision, but now the speed is

$$v^2 = v_x^2 + v_y^2$$

As an example, let us consider a two dimensional problem in which a particle of mass m_1 collides with a particle of mass m_2 , initially at rest. After the collision, m_1 moves at an angle q with respect to the *x*-axis and m_2 moves at an angle f with respect to the *x*-axis.

Example 9.7 A 1500-kg car traveling east with 25 m/s collides at an intersection with a 2500-kg van traveling north with of 20 m/s as shown. Find the final velocity after collision, if both cars moved together after collision.



Solution Conservation of momentum in x-direction, gives

$$m_1 v_{1i} + m_2(0) = (m_1 + m_2) v_f \cos q \Rightarrow$$

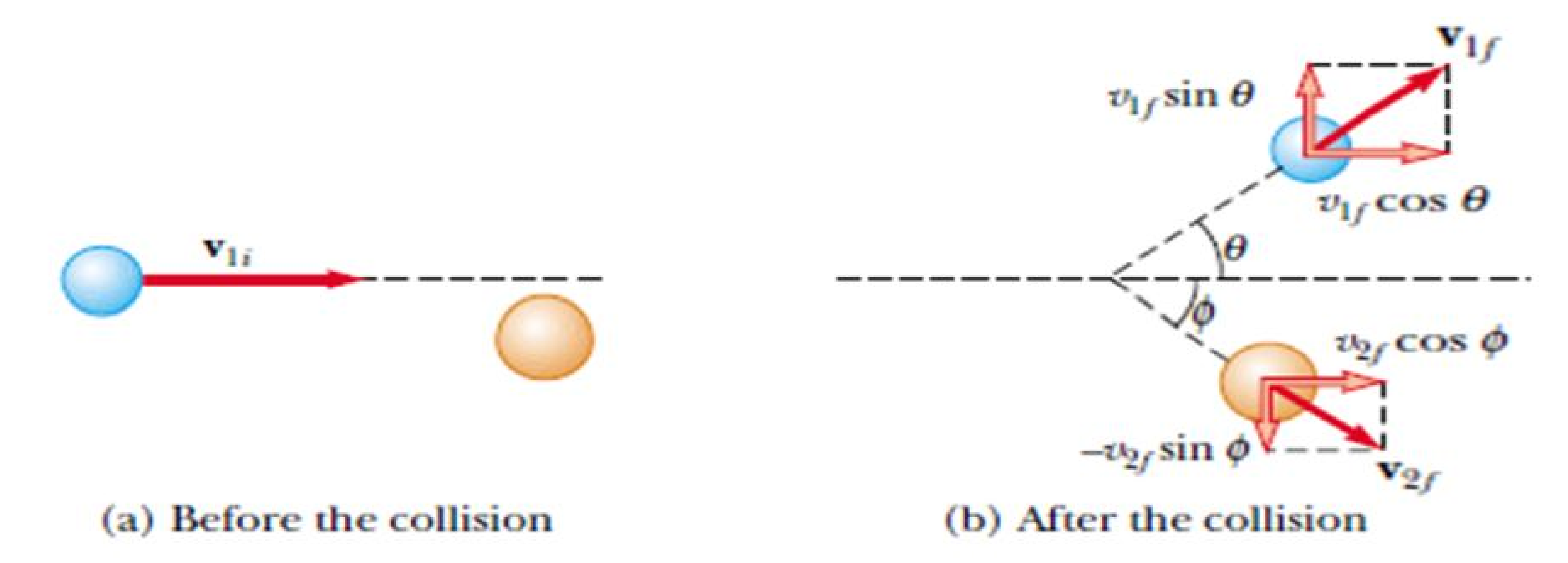
$$1500 \times 25 = (1500 + 2500) v_f \cos q \Rightarrow 37500 = 4000 v_f \cos q \qquad (1)$$

And in y-direction, gives $m_1(0) + m_2 v_{2i} = (m_1 + m_2) v_f \sin q \Rightarrow$

$$2500 \times 20 = (1500 + 2500) v_f \sin q \Rightarrow 5000 = 4000 v_f \sin q \qquad (2)$$

Dividing Eq. (2) by Eq.(1) $\Rightarrow \frac{50000}{37500} = \tan q \Rightarrow q = \tan^{-1}(1.33) = 53.1^{\circ}$

Substituting in Eq. (2) $\Rightarrow v_f = \frac{50}{4 \sin 53.1^{o}} = 15.6\,\text{m/s}$



(a) Before the collision (b) After the collision

Example 9.8 A particle of mass m and initial velocity v_{1i}=3.5×10^5 m/s collides, elastically, with another particle with equal mass m, and initially at rest. After collision one particle moves at an angle q=37^o above the x-axis, while the other particle moves at an angle f below the x-axis. Find the final velocities and the angle f.

Solution Conservation of momentum in x-direction, gives

$$mv_{1i} = mv_{1f} \cos q + mv_{2f} \cos f \quad \Rightarrow v_{1i} = v_{1f} \cos q + v_{2f} \cos f \qquad (1)$$

And in y-direction, gives

$$0 = mv_{1f} \sin q - mv_{2f} \sin f \quad \Rightarrow \; 0 = v_{1f} \sin q - v_{2f} \sin f \qquad (2)$$

The conservation of kinetic energy gives

$$\tfrac{1}{2} m v_{1i}^2 = \tfrac{1}{2} m v_{1f}^2 + \tfrac{1}{2} m v_{2f}^2 \quad \Rightarrow \quad v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \qquad (3)$$

Squaring the two sides of equations (1) and (2) gives, respectively

$$v_{1\mathrm{i}}^2 = v_{1\mathrm{f}}^2 \cos^2 q + v_{2\mathrm{f}}^2 \cos^2 f + 2 v_{1\mathrm{f}} v_{2\mathrm{f}} \cos q \cos f \qquad (4)$$

$$0 = v_{1\mathrm{f}}^2 \sin^2 q + v_{2\mathrm{f}}^2 \sin^2 f - 2 v_{1\mathrm{f}} v_{2\mathrm{f}} \sin q \sin f \qquad (5)$$

Now adding equations (4) and (5) gives

$$v_{1\mathrm{i}}^2 = v_{1\mathrm{f}}^2 + v_{2\mathrm{f}}^2 + 2 v_{1\mathrm{f}} v_{2\mathrm{f}} [\cos(q + f)] \qquad (6)$$

Subtracting Equation (3) from Equation (6) gives

$$0 = 2 v_{1\mathrm{f}} v_{2\mathrm{f}} [\cos(q + f)] \quad \Rightarrow \cos(q + f) = 0 \quad \Rightarrow \quad q + f = 90^o \Rightarrow f = 53^o$$

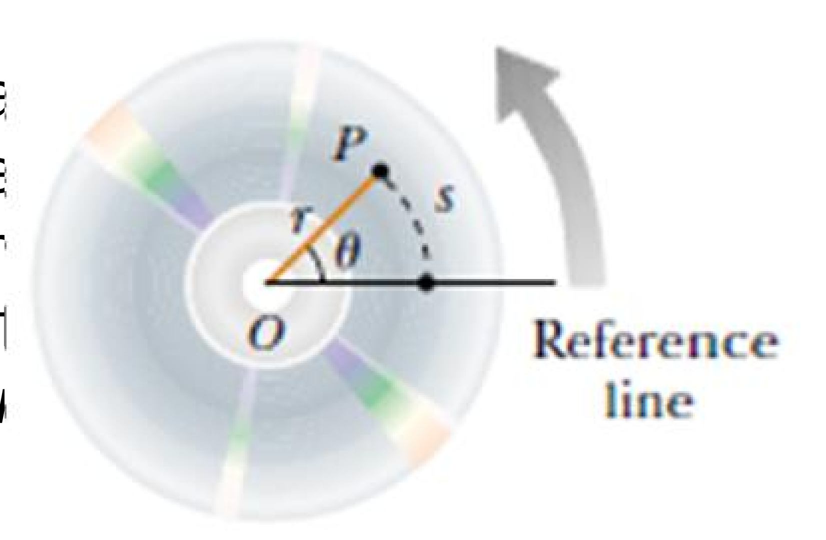
Substituting the values of q, f, and $v_{1\mathrm{i}}$ into Equations (1) and (2) and solving the two equations for $v_{1\mathrm{f}}$ and $v_{2\mathrm{f}}$ we obtain

$$v_{1\mathrm{f}} = 2.8 \times 10^5 \text{ m/s} \qquad v_{2\mathrm{f}} = 2.1 \times 10^5 \text{ m/s}$$

CHAPTER 10
ROTATIONAL MOTION

10.1 ANGULAR VELOCITY

Consider a rigid body rotates about a fixed axis through point *O* in *x*-*y* plane as shown. Any particle at point *P* in this rigid body rotates in a circle of radius *r* about *O***.** The angle between *r* and the positive *x*-axis is q .

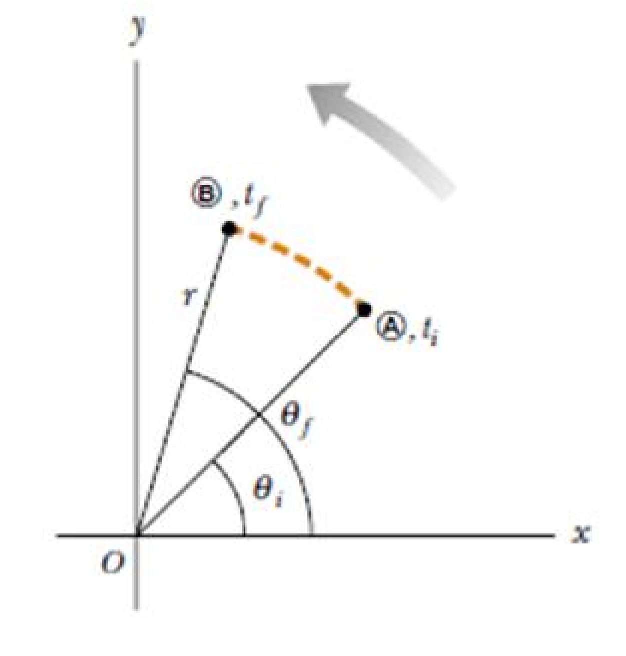


As the point *P* moves around the circle the angle q changes while *r* remains constant. Thus q serves as a coordinate to describe the rotational position of a particle. The angle is related to the displacement (arc length) *s*, through the relation

$$s = rq$$

Here the unit of q is radian with $1 rad. = \frac{360^{0}}{2p} \approx 57.3^{0}$

In analogy to the linear velocity, the **average angular velocity** of a particle moves from point *A* to point *B* in the figure is defined as



$$\overline{w} = \frac{q_f - q_i}{t_f - t_i} = \frac{\Delta q}{\Delta t}$$



The **instantaneous angular velocity** is defined as

$$w = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

As it is clear from the above equations, the unit of w is rad./s, or simply s^{-1} . The direction of w is along the axis of rotation with its sense can be determined from the **right hand rule:** turn the four fingers of your right hand with the direction of rotation; your thumb then gives the direction of .

10.2 ANGULAR ACCELERATION

The **average angular acceleration** of a particle is defined as

$$\bar{a} = \frac{w_f - w_i}{t_f - t_i} = \frac{\Delta w}{\Delta t}$$

And the **instantaneous angular acceleration** is defined as

$$a = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

The unit of a is rad./s^2 , or simply s^{-2} .

10.3 ROTATION WITH CONSTANT a

$$a = \frac{dw}{dt} \quad \Rightarrow \quad dw = a\,dt \quad \Rightarrow \quad \int_{w_o}^{w} dw = a \int_0^t dt \quad \Rightarrow$$

$$\boxed{w = w_o + at}$$

$$\text{Now } w = \frac{dq}{dt} = w_o + at \quad \Rightarrow \quad dq = (w_o + at)dt \quad \Rightarrow$$

$$dq = (w_o + at)dt \quad \Rightarrow \quad \int_{q_o}^{q} dq = \int_{0}^{t} (w_o + at)dt \quad \Rightarrow$$

$$\boxed{q = q_O + w_O t + \frac{1}{2}at^2}$$

Eliminating t from the 2-equations we get

$$\boxed{w^2 = w_O^2 + 2a(q - q_O)}$$

These three equations are of the same forms as those for linear motion with the replacing:

$$x \rightarrow q, \qquad v \rightarrow w, \text{ and } \qquad a \rightarrow a$$

Example 10.1 A wheel rotates with constant angular acceleration of 3.5 rad/s^2 . If the angular velocity of the wheel is 2 rad/s at t = 0.
a) What angle does the wheel rotate through in 2 s?
b) What is the angular velocity at t = 2 s?

Solution Using the equation

$$q = q_O + w_O t + \frac{1}{2} a t^2 \quad \Rightarrow$$

$$q = 2 \times 2 + \frac{1}{2} \times 3.5 \times 4 = 11 \text{rad}$$

Now Using the equation

$$w = w_o + a t \quad \Rightarrow$$

$$w = 2 + 3.5 \times 2 = 9 \text{ rad/s}$$

10.4 RELATIONSHIPS BETWEEN ANGULAR AND LINEAR VARIABLES

It is known that

v $\quad w$

$$s = rq \quad \Rightarrow \quad \frac{ds}{dt} = r\frac{dq}{dt} \quad \Rightarrow \quad \boxed{v = rw}$$

The tangential acceleration is defined as

$$a_q = \frac{dv}{dt} = r\frac{dw}{dt} \quad \Rightarrow \quad a_q = ra$$

a

While the radial acceleration is defined as $\quad a_r = \frac{v^2}{r} = rw^2$

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2 a^2 + r^2 w^4} \quad \Rightarrow \quad a = r\sqrt{a^2 + w^4}$$

Remark: For a rigid body rotating about a fixed axis, every particle on the body has the same angular velocity and the same angular acceleration, while the linear velocity and the linear acceleration differ from point to point.

Example 10.2 A wheel of radius 30 cm rotates about a fixed axis with initial angular velocity 150 rev/min and takes 20 s to come to rest.

a) Calculate the angular acceleration.

b) How many rotations does the wheel make before coming to rest?

c) Calculate the radial and tangential accelerations at *t* = 0.

Solution a) Noting that each revolution represents 2π radians we have for the initial angular velocity

$$w_o = \frac{150(2p)}{60} = 15.7\,\text{rad/s}$$

Using the equation $w = w_0 + at \quad \Rightarrow \quad a = \frac{-w_o}{t} = \frac{-5p}{20} = -0.79\,\text{rad/s}^2$

b) Now using the equation

$$q = q_O + w_O t + \tfrac{1}{2} a t^2 \quad \Rightarrow$$

$$q = 5p \times 20 - \tfrac{1}{2} \times \frac{p}{4} (20)^2 = 50p \quad \Rightarrow$$

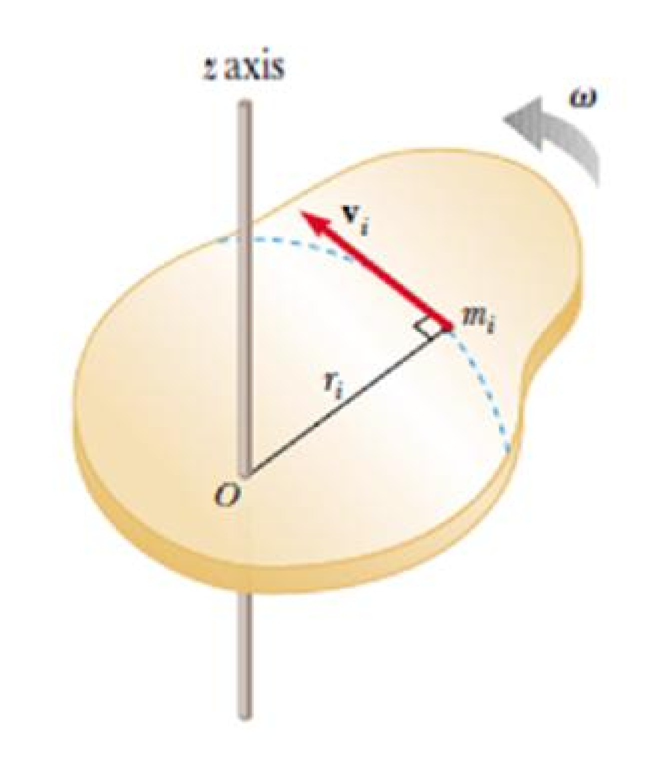
$$\text{No. of Rev.} = \frac{q}{2p} = 25\,\text{Rev}$$

c) Using the equations

$$a_\mathrm{r} = r w_o^2 = 0.3(5p)^2 = 7.5p^2\,\mathrm{m/s}^2$$

$$a_\mathrm{t} = r a = 0.3(-\frac{p}{4})\,\mathrm{m/s}^2$$

10.5 ROTATIONAL KINETIC ENERGY



Consider a rigid body consists of small particles, rotates about fixed axis with angular velocity ω.

The kinetic energy of the *i* th particle of mass m_i and linear speed v_i is given by

$$K_i = \frac{1}{2} m_i v_i^2$$

But $v_i = r_i w \quad \Rightarrow$

$$K_i = \frac{1}{2} m_i r_i^2 w^2$$

The total kinetic energy of the rotating rigid body is the sum of the kinetic energies of the individual particles, i.e.,

$$K = \sum_i K_i = \frac{1}{2}\left(\sum_i m_i r_i^2\right) w^2$$

$$\text{or } K = \frac{1}{2} I w^2$$

$$\text{with } I = \left(\sum_i m_i r_i^2 \right)$$

is called the **moment of inertia.** It represents the mass in all rotational equations.

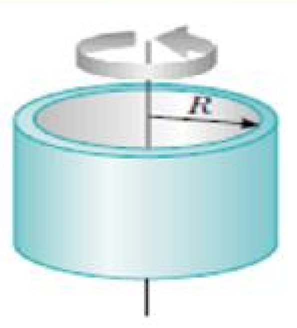
For a body of continuous mass distribution, the summation in Equation can be replaced by an integration over the body, i.e.,

$$I = \int r^2 dm$$

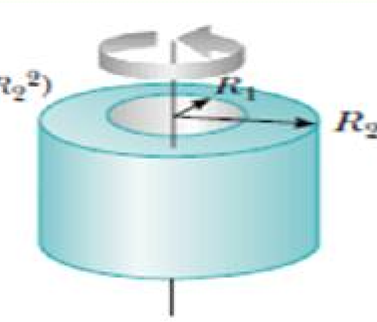
where *r* is the perpendicular distance from the element *dm* to the axis of rotation.

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

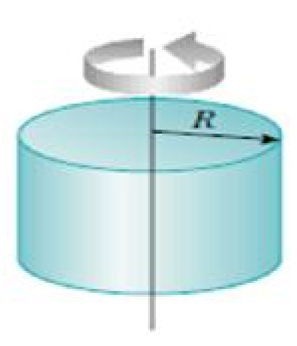
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



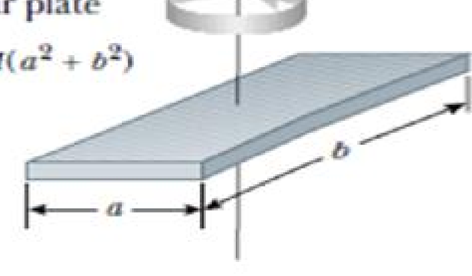
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$



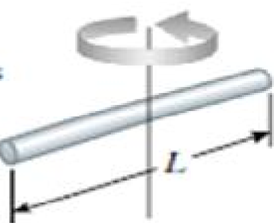
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



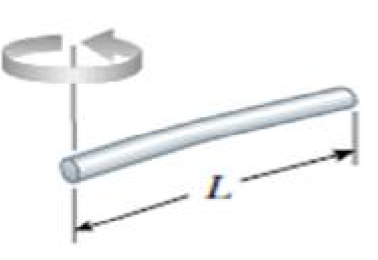
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$

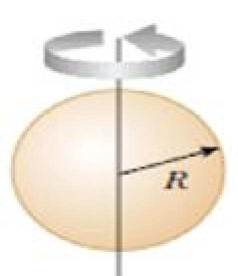
Long thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$



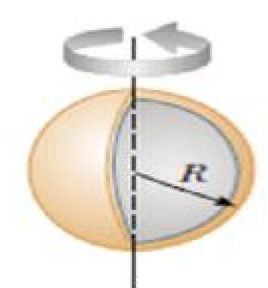
Long thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$



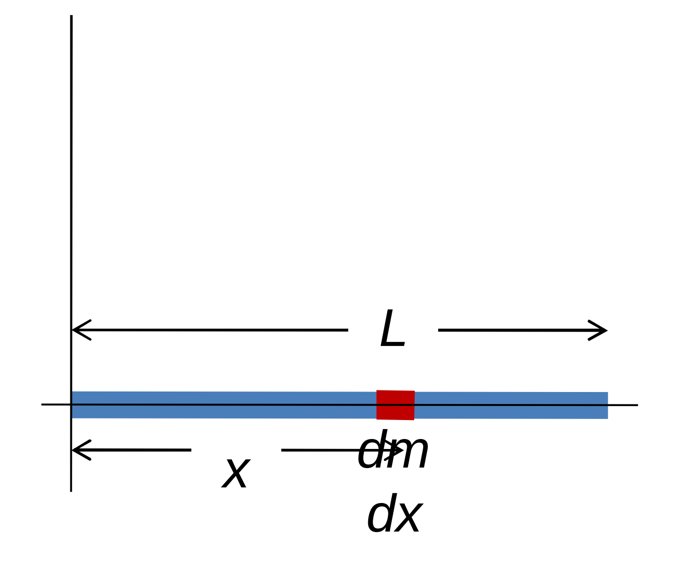
Solid sphere
$I_{CM} = \frac{2}{5}MR^2$



Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$



Example 10.3 Calculate the moment of inertia of a uniform, thin rod of mass *M* and length *L* about an axis normal to its length
a) at one end,
b) at its center.



Solution **a)** Let us divide the rod into small elements each of mass *dm*.

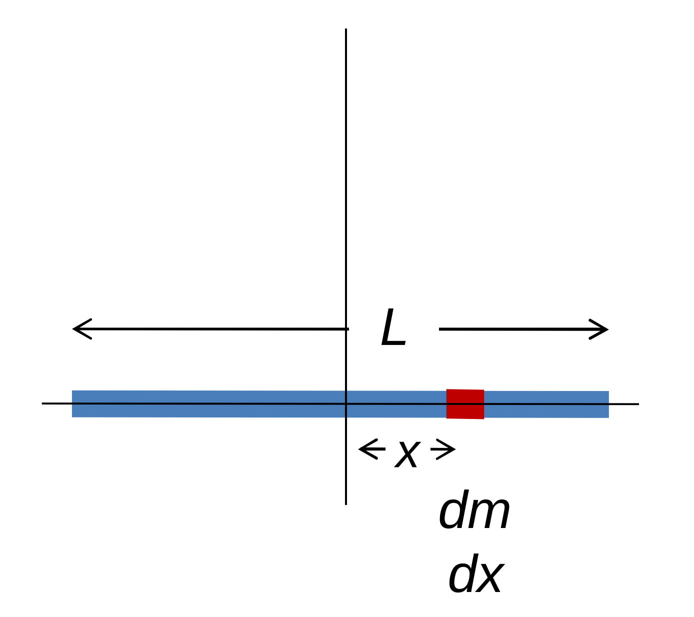
Using the equation $I = \int r^2 dm \quad \Rightarrow \quad I = \int x^2 dm$

Using the definition of linear mass density λ, we have

$$\lambda = \frac{dm}{dx} = \frac{M}{L} \quad \Rightarrow \quad dm = \lambda\, dx$$

$$I = \lambda \int_0^L x^2 dx = \lambda \left[\tfrac{1}{3} x^3\right]_0^L = \tfrac{1}{3} \lambda L^3 \quad \Rightarrow \quad I = \tfrac{1}{3} ML^2$$

b) Now if the axis of rotation is at the center of the rod we have again



$$I = \int r^2 dm \quad \Rightarrow \quad I = \int x^2 dm$$

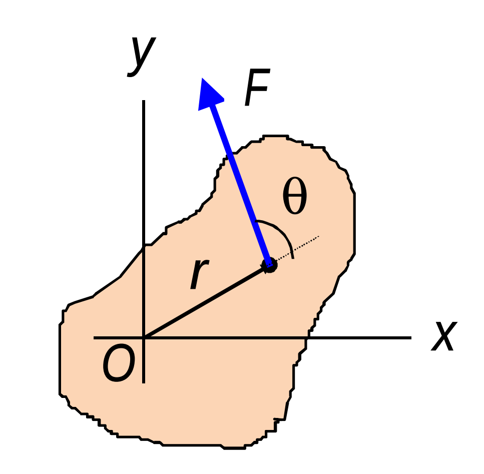
$$l = \frac{dm}{dx} = \frac{M}{L} \quad \Rightarrow \quad dm = l\,dx$$

$$I = l \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx \quad = l \left[\frac{1}{3} x^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12} l L^3 \quad \Rightarrow$$

$$I = \frac{1}{12} M L^2$$

10.6 TORQUE

Consider a force **F** acting on the particle *P* in a rigid body, as shown. The torque due to the force **F** is defined by the cross product of the vector **r** and the force vector **F,** displacement



$$t = \boldsymbol{r} \times \boldsymbol{F}$$

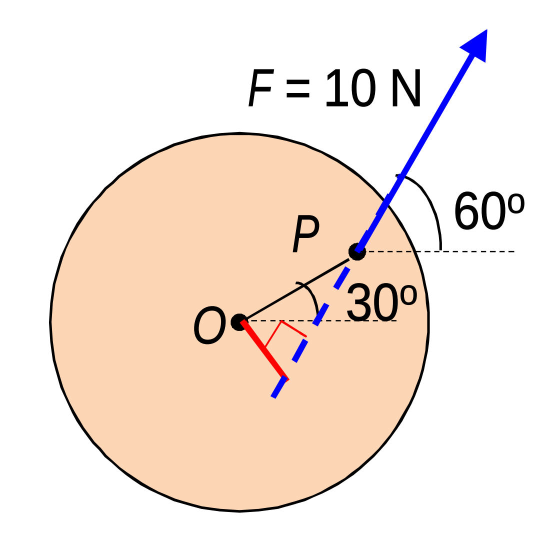
The torque is a vector quantity with magnitude

$$t = Fr\sin q = r_{\perp}F$$

with $r_{\perp} = r\sin q$ is called the moment arm of the force and is defined as the perpendicular distance from the point of rotation to the line of action of the force.

t is positive if it tends to rotate the body counterclockwise, in the direction of increasing θ. It is negative if it tends to rotate the body clockwise. From the 1st equation, the direction of t is perpendicular to the plane containing ***r*** and ***F*****.**

Example 10.4 A force F =10 N is applied at point P, 0.5 m from the center O of a wheel. Op makes an angle of 30 with x-axis and the force makes an angle of 60 with x-axis, as shown. Calculate the torque on the wheel.



Solution The arm of the force is given by

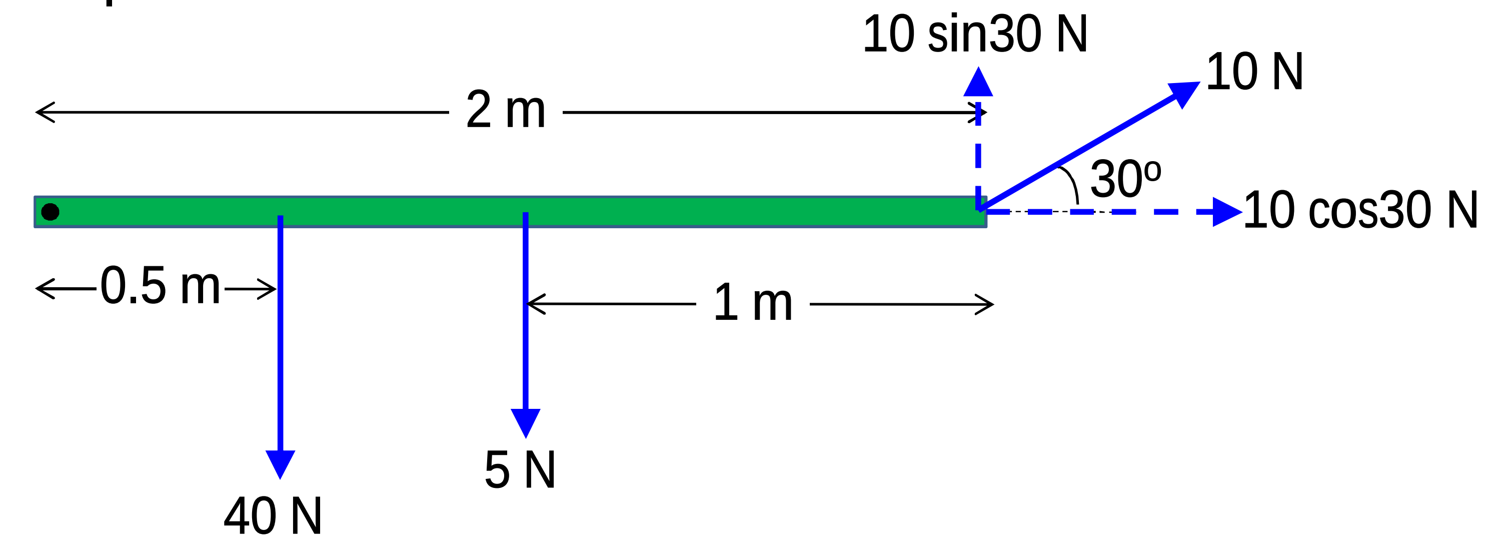
$$r_{\perp} = r \sin 30 = 0.5 \sin 30 = 0.25\,\text{m}$$

The torque of the force is then

$$t = r_{\perp} F = 10 \times 0.25 = 2.5\,\text{N.m}$$

The negative sign means that the rotation is clockwise.

Example 10.5 A rod of length L=2 m and weight 5 N is acted on by two forces as shown in Figure 10.8. Find the total torque acting on the rod about point O.

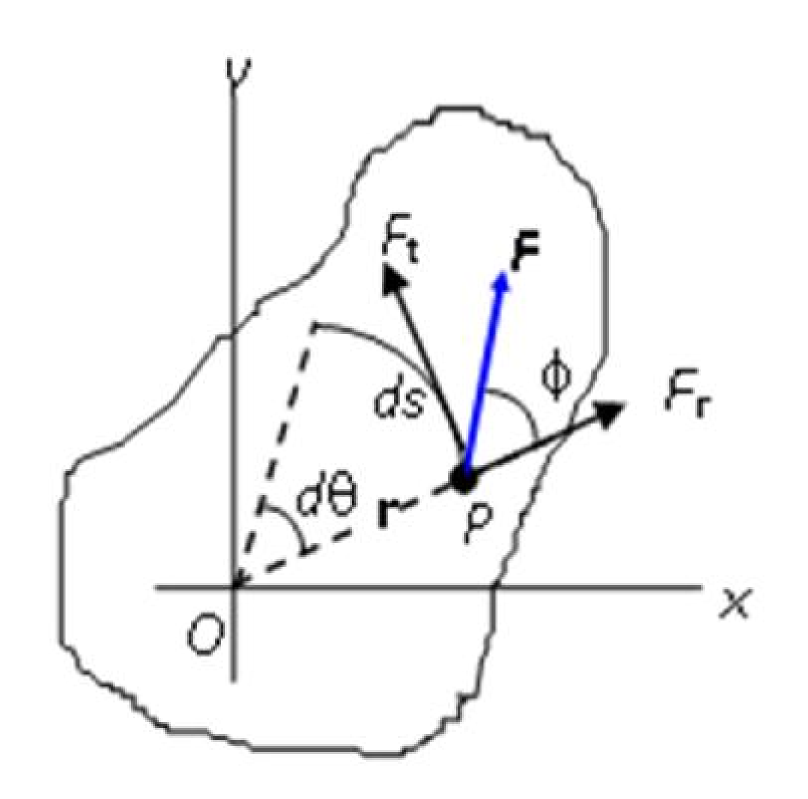


Solution Let us first resolve the 10-N force. Taking the +ve sense counterclockwise we get

$$t = -(40\times0.5)-(5\times1)+(2\times10\sin 30) = -15\,\text{N.m}$$

10.7 TORQUE AND ANGULAR ACCELERATION

Consider a force F acting on a rigid body free to rotate about *O*. During the infinitesimal time *dt*, the element *P* of mass *dm* will move an infinitesimal distance *ds* along a circular path of radius *r* as the body rotates through an infinitesimal angle *dq*, where



$$ds = rdq$$



The torque acting on the body is

$$t = F_t r$$

$$\text{But } F_t = M\left(a_{cm}\right)_t \quad \text{and} \quad M\left(a_{cm}\right)_t = \sum_{\text{i}=1}^{\text{n}} m_{\text{i}} a_{\text{ti}} \quad = \int a_t dm \quad \Rightarrow$$

$$t = \int r a_t dm \qquad \text{But } a_t = ra \qquad \Rightarrow$$

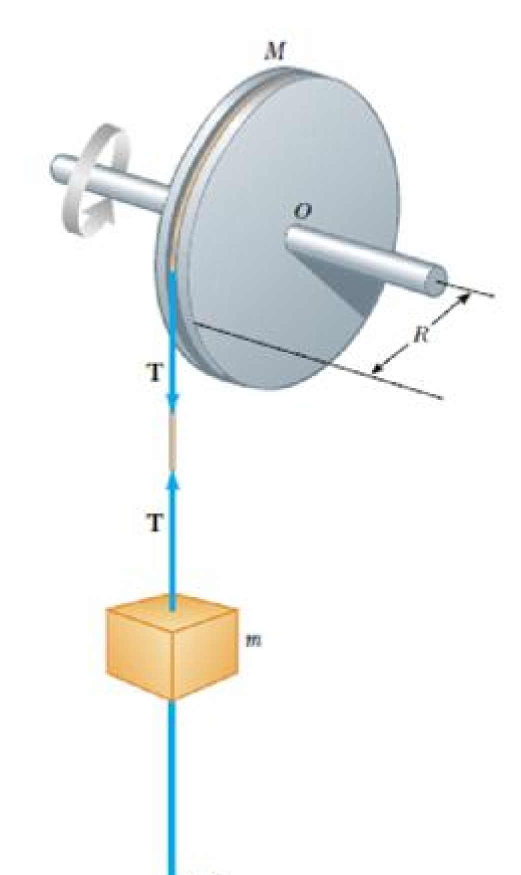
$$t = a\int r^2 dm = Ia$$

If there is more than one force doing torque on the body, the last equation can be generalized as

$$\boxed{\sum t = Ia}$$

The last equation can be considered as Newton’s second law for rotational motion.

Example 10.6 A wheel of radius *R*, mass *M*, and moment of inertia *I* is rotating on a frictionless, horizontal axle as shown. A light cord wrapped around the wheel and supports a body of mass *m*. Calculate the linear acceleration, *a,* of the body, the angular acceleration, α, of the wheel , and the tension, *T*, in the cord.



Solution For the wheel we apply

$$\sum t = Ia \quad \Rightarrow \quad TR = Ia \qquad (1)$$

Note that the weight of the wheel and the force of the axle on the wheel don't do any torque on the wheel. Why?

And for the mass we apply

$$\sum F = ma \quad \Rightarrow \quad mg - T = ma \qquad (2)$$

The third equation is obtained from the relation between the angular and linear acceleration:

$$a = Ra \qquad (3)$$

Using Eq. (1) & Eq.(3) $\Rightarrow$

$$T = I\frac{a}{R^2} \qquad (4)$$

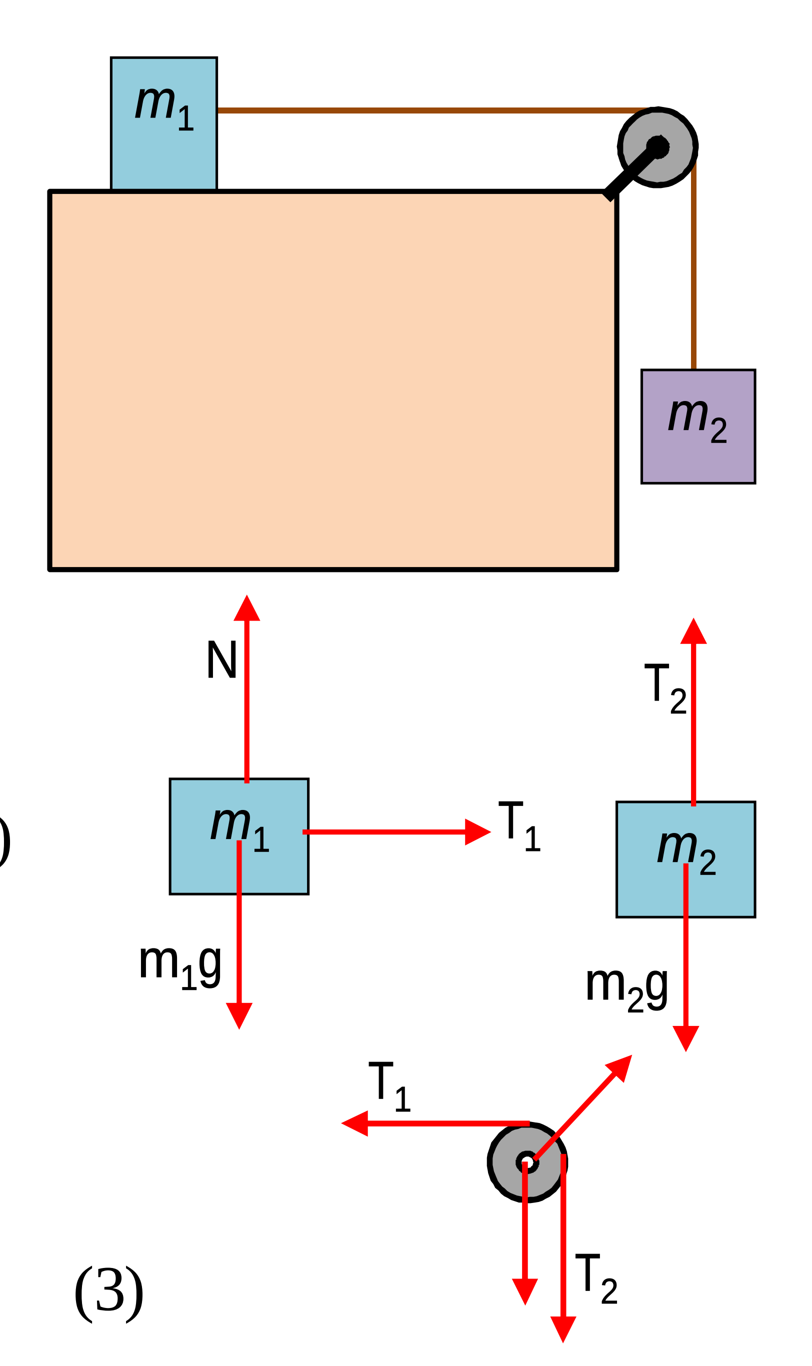
Now adding Eq. (2) & Eq.(4) $\Rightarrow$

$$mg = \left(m + \frac{I}{R^2}\right)a \qquad \Rightarrow \qquad a = \frac{mg}{\left(m + \frac{I}{R^2}\right)}$$

Substituting back in Eq.(4) $\Rightarrow$

$$T = \frac{mg}{\left(\frac{mR^2}{I} + 1\right)}$$

Example 10.7 Two blocks having masses m_1 *and* m_2 *are connected to each* other by a light cord that passes over a frictionless pulley, with moment of inertia *I and radius R,* as shown. Find the acceleration of each block and the tensions in the cord.



Solution The free-body diagrams of the 2-masses and the pulley are shown.

For the wheel we apply

$$\sum t = I a \quad \Rightarrow \quad (T_2 - T_1)R = I a \qquad (1)$$

And for m_1 we apply

$$\sum F_x = ma \quad \Rightarrow \quad T_1 = m_1 a \qquad (2)$$

And for m_2 we apply

$$\sum F_y = ma \quad \Rightarrow \quad m_2 g - T_2 = m_2 a \qquad (3)$$

The fourth equation is obtained from the relation between the angular and linear acceleration:

$$a = Ra \qquad (4)$$

Subtracting Eq. (1) & Eq.(2) $\Rightarrow$ $T_2 - T_1 = m_2 g - (m_1 + m_2)a \qquad (5)$

Substituting Eq. (5) into Eq.(1) $\Rightarrow$

$$m_2 g - (m_1 + m_2)a = \frac{Ia}{R} = \frac{Ia}{R^2} \quad \Rightarrow \quad a = \frac{m_2 g}{\left(\frac{I}{R^2} + m_1 + m_2\right)}$$

From Eq. (2) we get $T_1 = \dfrac{m_1 m_2 g}{\left(\frac{I}{R^2} + m_1 + m_2\right)}$

And from Eq. (3) we get

$$T_2 = m_2(g - a) = m_2 g\left(1 - \frac{m_2}{m_1 + m_2 + \frac{I}{R^2}}\right)$$

10.8 WORK AND ROTATIONAL MOTION

Let us consider again the situation in the figure shown

The work *dW* done by the force **F** is

$$dW = \mathbf{F}\cdot d\mathbf{s} = F\sin\phi\, ds = F\sin\phi\,(r\,d\theta)$$

$$\text{But } (F\sin\phi)r = F_t r = \tau \quad \Rightarrow$$

$$dW = \tau\, d\theta$$



The total work *W* is then

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

The power is now given as $P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$

Which is the rotational analogue of $P = Fv$

$$\text{Now } t = Ia = I\frac{dw}{dt} \quad \Rightarrow$$

$$P = \frac{dW}{dt} = tw = Iw\frac{dw}{dt} \quad \Rightarrow \quad dW = Iwdw \quad \Rightarrow$$

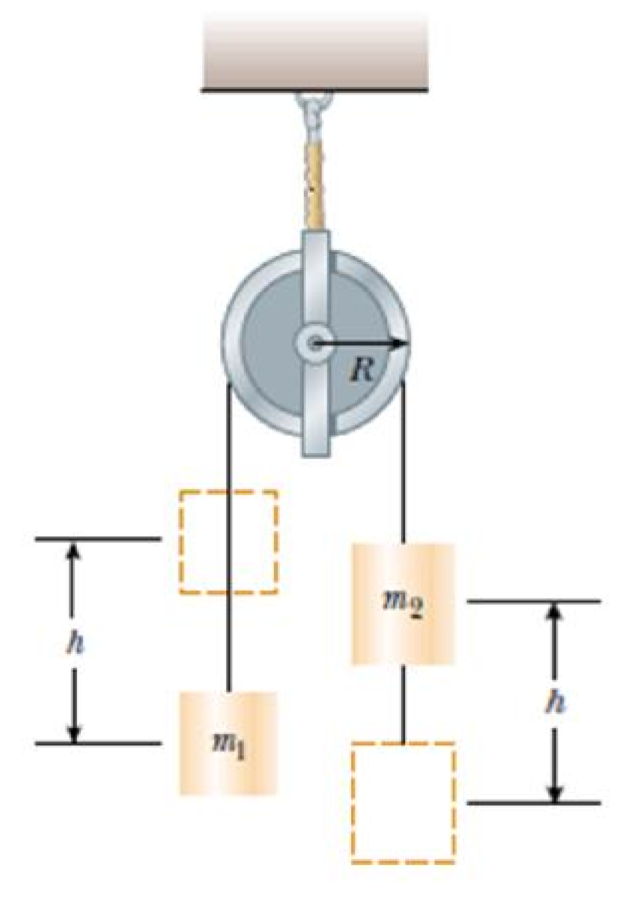
$$W = I\int_{w_o}^{w} w\,dw \quad = \tfrac{1}{2}I\left(w^2 - w_o^2\right) = \Delta K$$

$$\text{with } = K = \tfrac{1}{2}Iw^2$$

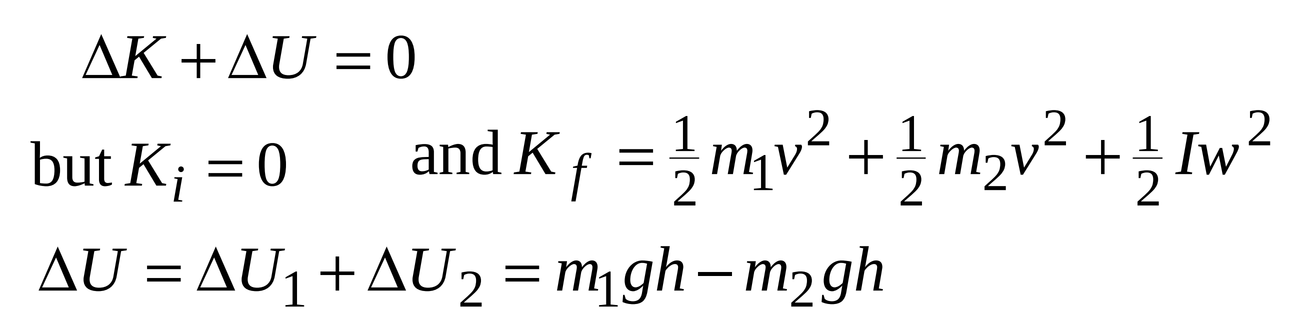
is the rotational kinetic energy. It is the rotational analogue of the linear equation

$$K = \tfrac{1}{2}mv^2$$

Example 10.8 Consider two masses connected by a string passing over a pulley having a moment of inertia *I* about its axis of rotation, as shown. The string does not slip on the pulley, and the system is released from rest. Find the linear velocity of the masses after m_2 descends through a distance *h*, and the angular velocity of the pulley at this time.



Solution Applying the conservation of mechanical energy principle we get



$$\Delta K + \Delta U = 0$$

$$\text{but } K_i = 0 \qquad \text{and } K_f = \tfrac{1}{2} m_1 v^2 + \tfrac{1}{2} m_2 v^2 + \tfrac{1}{2} I w^2$$

$$\Delta U = \Delta U_1 + \Delta U_2 = m_1 g h - m_2 g h$$

Knowing that $v = wr$

$$\frac{1}{2}m_1v^2+\frac{1}{2}m_2v^2+\frac{1}{2}\frac{I}{R^2}v^2+m_1gh-m_2gh=0 \quad \Rightarrow$$

$$v^2=\frac{2(m_1-m_2)gh}{m_1+m_2+\frac{I}{R^2}} \quad \Rightarrow$$

$$v=\sqrt{\frac{2(m_1-m_2)gh}{m_1+m_2+\frac{I}{R^2}}}$$

$$w=\frac{v}{R}=\sqrt{\frac{2(m_1-m_2)gh}{(m_1+m_2)R+\frac{I}{R}}}$$

10.9 ANGULAR MOMENTUM

The angular momentum ***L*** of the particle relative to an origin is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

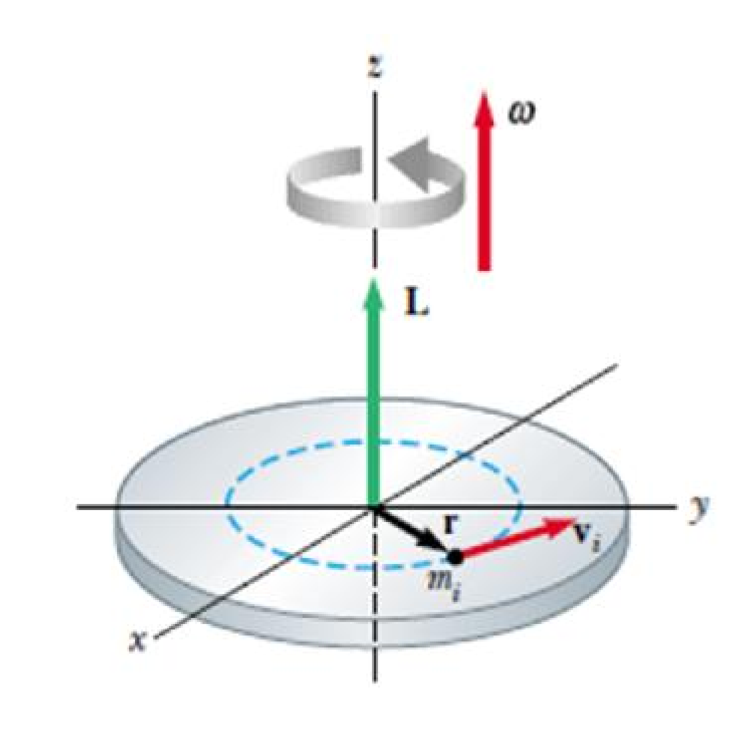
where ***r*** is the position vector of the particle relative to *O.* The angular momentum is a vector quantity with a direction perpendicular to the plane formed by ***r*** and ***v***, and its sense is determined by the right hand rule discussed in chapter 1.

The SI unit of ***L*** is kg.m^2/s

Consider a rigid body that rotates about the *z*-axis through the origin as shown.

The magnitude of the angular momentum of the particle m_i is

$$L_i = m_i v_i r_i = m_i w r_i^2$$



The total angular momentum of the body is, therefore

$$L = \sum_i L_i = w \sum_i m_i r_i^2 \quad \Rightarrow \quad L = Iw$$

Which is the rotational analogue of the linear equation $p = mv$

10.10 CONSERVATION OF ANGULAR MOMENTUM

It is known that $t = \boldsymbol{r} \times \boldsymbol{F}$ and $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \quad \Rightarrow \quad t = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$

$$\text{but } \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} \quad \Rightarrow$$

$$t = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) \quad \Rightarrow \quad t = \frac{d\boldsymbol{L}}{dt}$$

Which is the rotational analogue of the linear equation $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$

From the last equation, if the external torque acting on a system is zero, then ***L*** is constant, i.e., ***L*** is conserved, or

$$\boldsymbol{L}_i = \boldsymbol{L}_f \qquad \text{or } I_\text{i} w_\text{i} = I_\text{f} w_\text{f}$$

Example 10.9 Two disks one of mass 1 kg and radius 0.2 m, and the other of mass 4 kg and radius 0.15 m. The lighter disk is rotating with initial angular velocity of 80rad/s, while the heavier disk rotates with initial angular velocity of 240 rad/s. If the two disks are pushed into contact, find the common angular velocity of the combination.

Solution The moment of inertia of the two disks are

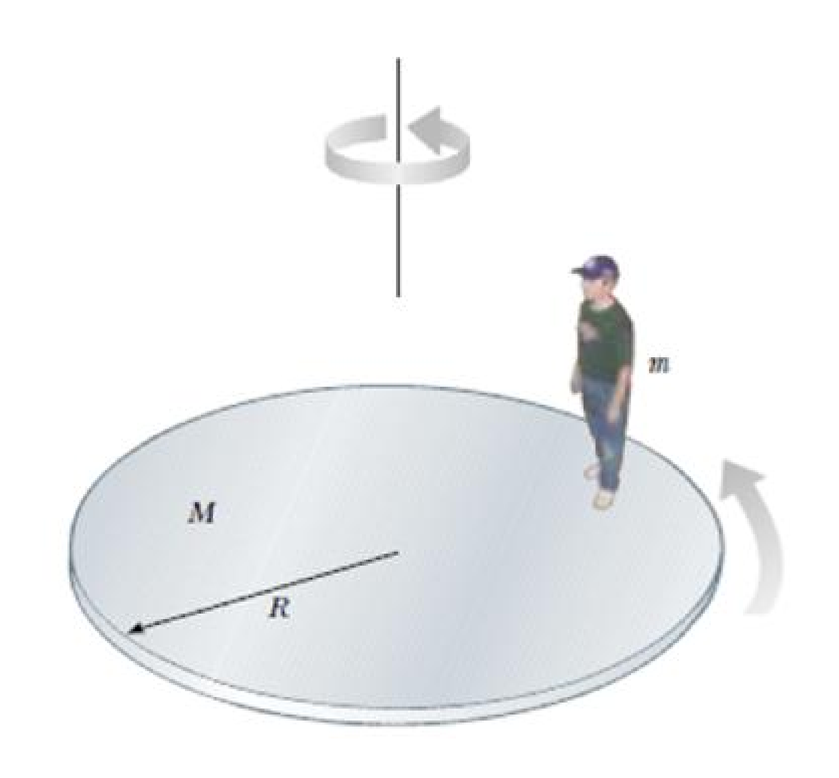
$$I_1 = \tfrac{1}{2} m_1 r_1^2 = \tfrac{1}{2}(1)(0.2)^2 = 0.02\,\text{kg.m}^2$$

$$I_2 = \tfrac{1}{2} m_2 r_2^2 = \tfrac{1}{2}(4)(0.15)^2 = 0.045\,\text{kg.m}^2$$

$$\text{Now } I_1 w_1 + I_2 w_2 = (I_1 + I_2) w_\text{f} \qquad \Rightarrow (0.02)(80) + (0.045)(240) = (0.065) w$$

$$\Rightarrow \quad w = 190.8\,\text{rad/s}$$

Example 10.10 A turntable rotates in a horizontal plane about a fixed, vertical axis, making one revolution in 15 s. The moment of inertia of the table about the axis of rotation is 1000 kg.m^2. A man of mass 70 kg, initially standing at the center of the turntable, starts to walk a way from the center along a radius. What is the angular velocity of the table when the man is 2 m from the center?



Solution Let us call the moment of inertia of the table I_t and the moment of inertia of the man I_m. The initial moment of inertia of the system is, therefore

$$I_i = (I_t + I_m)_i = (1000 + 0) = 1000\ \text{kg.m}^2$$

When the man has walked to the position r = 2 m, the final moment of inertia of the system becomes

$$I_f = \left(I_t + mr^2\right)_f = (1000 + 70 \times 4) = 1280 \text{ kg.m}^2$$

Applying the conservation of angular momentum law:

$$I_i w_i = I_f w_f \qquad \Rightarrow \qquad w_f = \frac{I_i}{I_f} w_i$$

$$\text{But } w_i = \frac{2p}{T} = \frac{2p}{15} = 0.42 \text{ rad/s} \qquad \Rightarrow$$

$$w_f = \left(\frac{1000}{1280}\right) 0.42 = 0.33 \text{ rad/s}$$

Linear Motion	Rotational Motion
x ® *q*, *v* ® *w*, *a* ® *a*, *m* ® *I*, *F*® *t*, *p* → L	
$v = \frac{dx}{dt}$ $a = \frac{dv}{dt}$	$w = \frac{dq}{dt}$ $a = \frac{dw}{dt}$
If a is constant $\begin{cases} v = v_o + at \\ x = v_o t + \frac{1}{2} a t^2 \\ v^2 = v_o^2 + 2ax \end{cases}$	If α is constant $\begin{cases} w = w_o + at \\ q = w_o t + \frac{1}{2} a t^2 \\ w^2 = w_o^2 + 2aq \end{cases}$
$\sum F = ma$	$\sum t = Ia$
$W = \int_i^f F dx$	$W = \int_i^f t \, dq$
$K = \frac{1}{2} m v^2$	$K = \frac{1}{2} I w^2$
$\wp = Fv$	$\wp = tw$
$p = mv$	$L = Iw$
$\sum F = \frac{dp}{dt}$	$\sum t = \frac{dL}{dt}$

CHAPTER 11
STATIC EQUILIBRIUM

11.1 EQUILIBRIUM OF A RIGID BODY

A rigid body is said to be in equilibrium when the two following conditions are satisfied:

1. (**Translational Equilibrium**) The resultant external force acting on the body must equal zero, i.e.,

$$\sum \boldsymbol{F} = 0 \quad \Rightarrow \quad \sum F_x = 0 \qquad \sum F_y = 0$$

Rotational Equilibrium) The resultant external torque must be zero about any origin, i.e.,

$$\sum t = 0$$

11.2 CENTER OF GRAVITY

It is a point in which all of the weight can be considered as being concentrated at that point.

For homogeneous, symmetric objects the center of gravity coincides with the geometric center of the object.

Problem solving strategy:

(i) Make a sketch of the system under consideration.

(ii) Draw a free-body diagram for the system showing all the external forces acting on it. Try to label each of these forces. Note that the internal forces acting from one part of the system on the others must not included in the diagram

(iii) Resolve all forces into their rectangular components, choosing an appropriate coordinate system

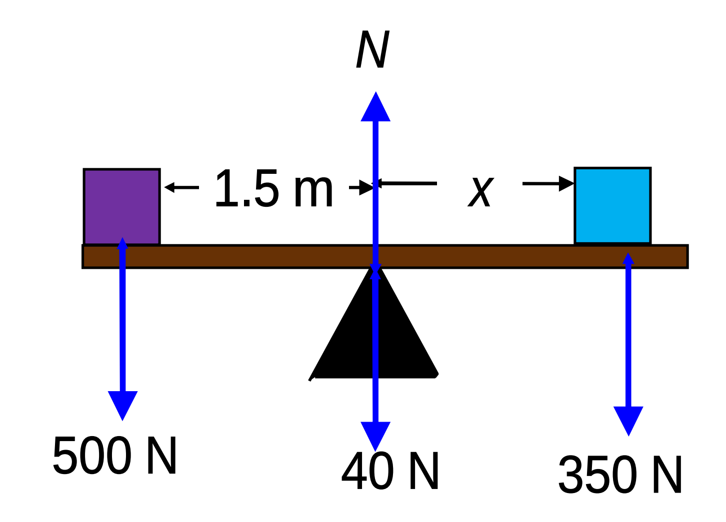
(iv) Choose a convenient origin for calculating the net torque on the object. Remember that the choice of the origin for the torque equation is arbitrary; therefore choose an origin that will simplify your calculation as much as possible.

(v) Applying the equations of equilibrium you get a set of linear equations with several unknowns. Simultaneous solutions of these equations give us the unknowns.

Example 11.1 A uniform rod of weight 40 N supports two masses weighing 500 N and 350 N, respectively, as shown. The support is under the center of gravity of the rod and the 500 N weight is 1.5 m from the center.

a) Determine the upward force N exerted on the rod by the support,

b) Determine where the 350 N mass should be put to balance the system.



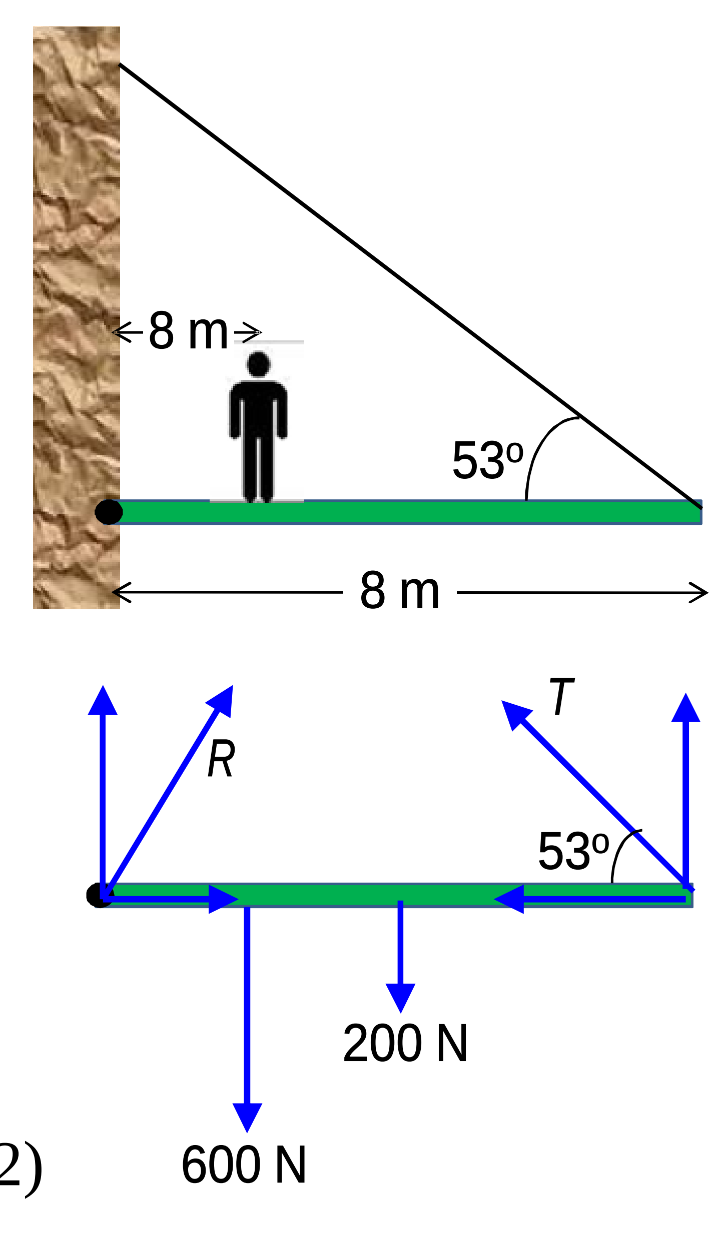
Solution All the forces acting on the system (the rod plus the two masses) are shown. Note that the forces acting by the rod on the two masses are not included, why? Now using

$$\sum F_y = 0 \quad \Rightarrow \quad N - 500 - 40 - 350 = 0 \quad \Rightarrow \quad N = 890\,\mathrm{N}$$

Now finding the torque about the center of gravity and using

$$\sum t = 0 \quad \Rightarrow \quad 500 \times 1.5 - 350 \times x = 0 \quad \Rightarrow \quad x = 2.14\,\mathrm{m}$$

Example 11.2 A uniform horizontal beam of length 8 m and weight 200 N is attached to a wall by a pin connection that allows the beam to rotate. Its far end is supported by a cable that makes an angle of 53 with horizontal. If a 600 N person stands 2 m from the wall, find the tension, *T*, in the cable and the force, *R*, exerted on the beam by the wall



Solution The free-body diagram of the system is shown. Applying the eq.

$$\sum F_x = 0 \quad \Rightarrow \quad R_x = T\cos 53^0 \qquad (1)$$

$$\sum F_y = 0 \quad \Rightarrow \quad R_y + T\sin 53^0 = 800N \qquad (2)$$

$$\sum t_o = 0 \quad \Rightarrow \quad T\sin 53^0 \times 8 - 600\times 2 - 200\times 4 = 0 \quad \Rightarrow$$

$$6.4T - 2000 = 0 \qquad (3)$$

From Eq.(3) we get

$$T = 313\,\text{N}$$

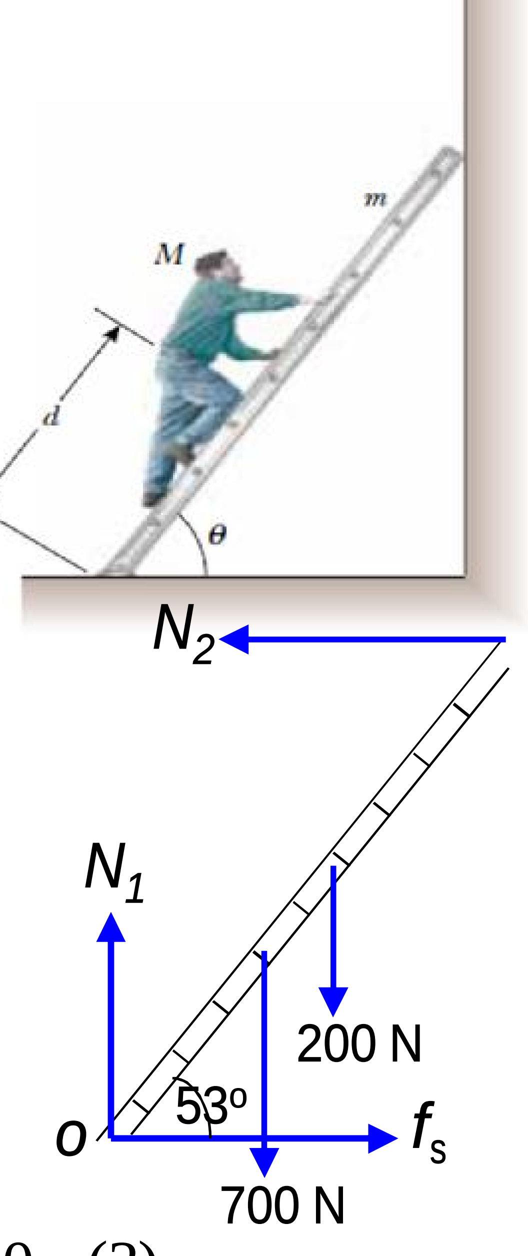
From Eq.(1) we get

$$R_x = 188\ \text{N}$$

From Eq.(2) we get

$$R_x = 550\,\text{N}$$

Example 11.3 A uniform ladder of length 6 m and weigh 200 N leans against a smooth vertical wall as shown. The ladder makes an angle of 53º with the ground and the coefficient of static friction between the ladder and the ground is 0.5. How far a man weighing 700 N can climb up the ladder before it starts to slip.



Solution The free-body diagram of the system is shown. Applying the eq.

$$\sum F_x = 0 \quad \Rightarrow \qquad f_s = N_2 \qquad (1)$$

$$\sum F_y = 0 \quad \Rightarrow \qquad N_1 - 200 - 700 = 0 \qquad (2)$$

$$\sum t_o = 0 \quad \Rightarrow$$

$$700(d\cos 53^o) + 200(3\cos 53^o) - N_2(6\sin 53^o) = 0 \quad (3)$$

When the ladder is on the verge of slipping, the frictional force is a maximum, i.e.,

$$f_s = m_s N_1 \qquad (4)$$

From Eq.(2) we have $N_1 = 900\,\mathrm{N} \quad \Rightarrow$

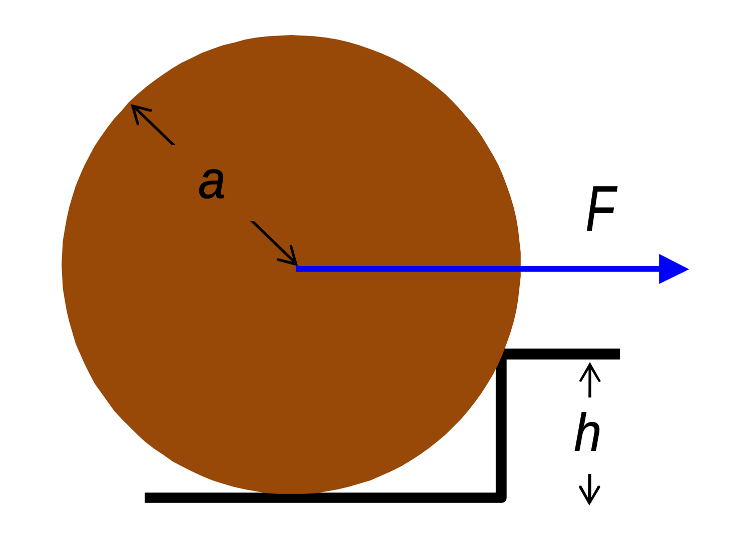
$$f_s = m_s N_1 = 0.5 \times 900 = 450\,\mathrm{N}$$

From Eq.(1) we have $N_2 = f_s = 450\,\mathrm{N}$

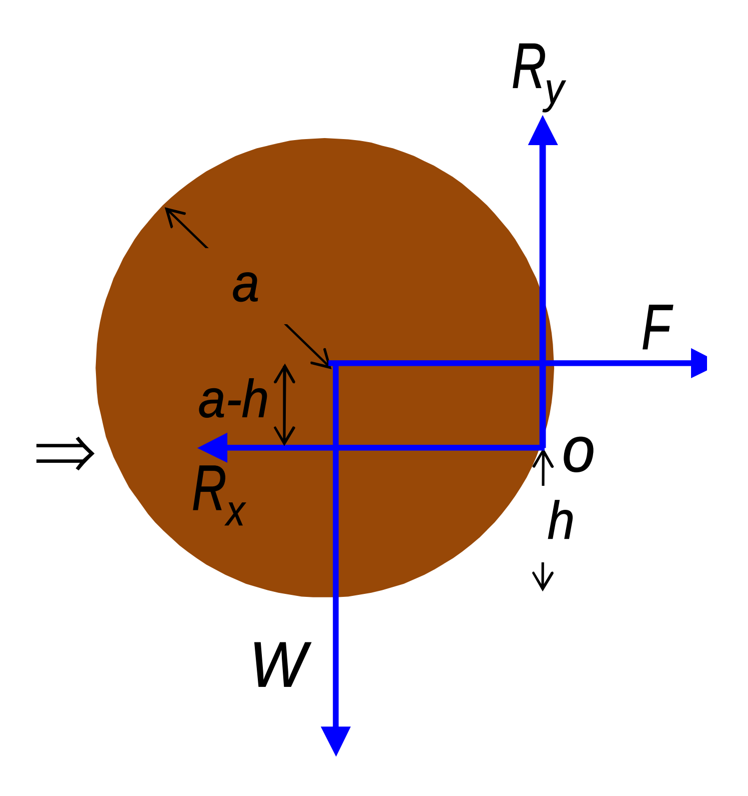
From Eq.(3) we have

$$d = \frac{(450)(6\sin 53^{\mathrm{o}}) - (200)(3\cos 53^{\mathrm{o}})}{700\cos 53^{\mathrm{o}}} = 4.26\,\mathrm{m}$$

Example 11.4 Find the force F applied horizontally at the axle of the wheel that is necessary to raise the wheel over a step of height h =0.2 m. The weight of the wheel is W=400N, and its radius is a =0.8 m.



Solution The free-body diagram of the system is shown. Because the wheel is in the verge of leaving the ground, there is no reaction force acting on the wheel by the ground. Applying the eq.



$$\sum t_o = 0 \quad \Rightarrow W\sqrt{a^2-(a-h)^2} - F(a-h) = 0 \quad \Rightarrow$$

$$F = \frac{W\sqrt{a^2-(a-h)^2}}{(a-h)} = 353 \text{ N}$$

11.3 ELASTIC PROPERTIES OF SOLIDS

In reality, all objects are deformable under the influence of external forces.

Stress: is the external force per unit cross-sectional area acting on an object.

Strain: is a measure of the degree of deformation.

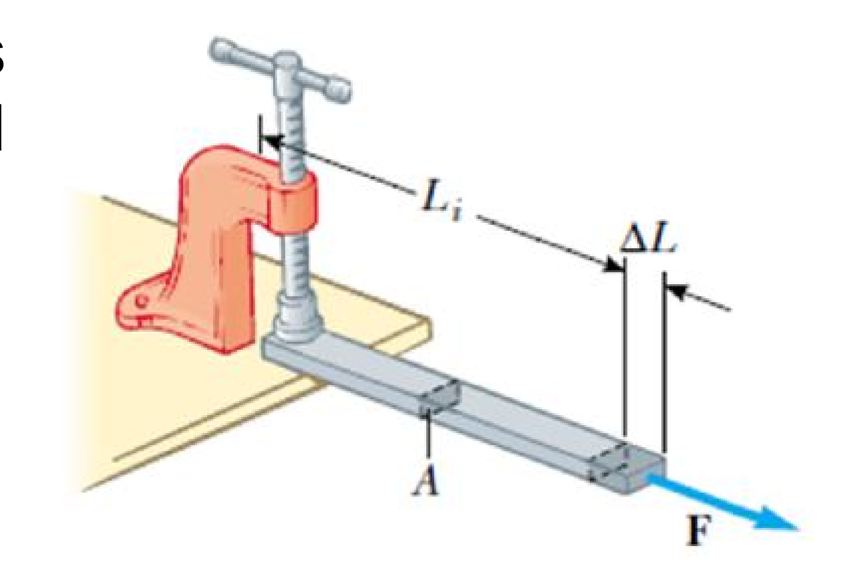
It is found that, for sufficiently small stress, the stress is proportional to the strain; the constant of proportionality is called the ***elastic modulus*, *i.e*,**

$$Elastic\ Modulus = \frac{Stress}{Srain}$$

We shall consider three types of deformation and define an elastic modulus for each:

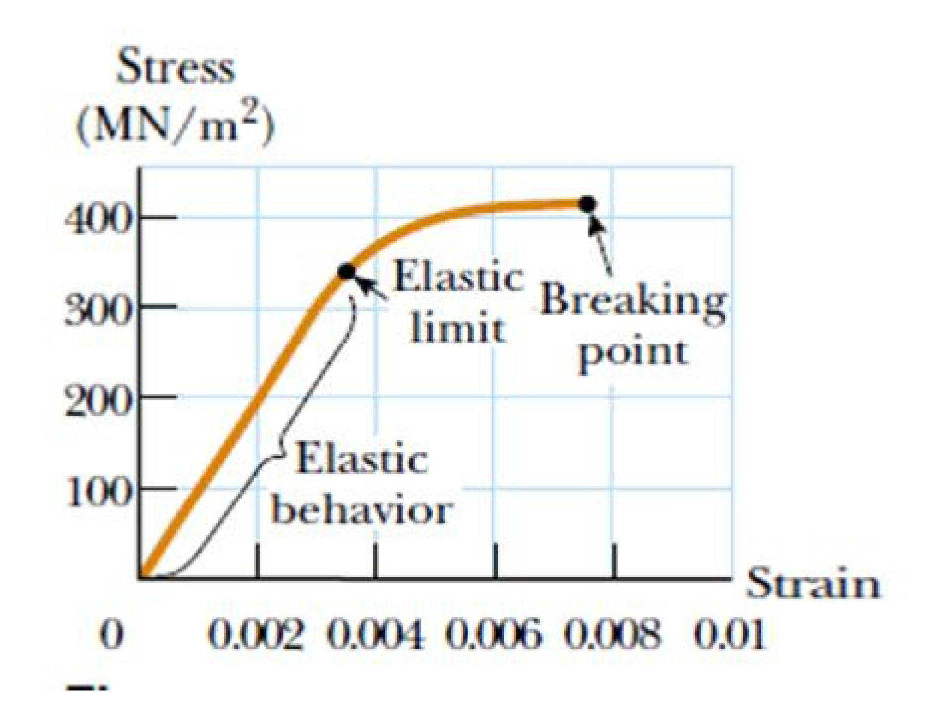
Young’s Modulus: Young's modulus measures the resistance of a solid to a change in its length, i.e.,

$$Y = \frac{tensile\ stress}{tensile\ strain} = \frac{F/A}{\Delta L/L}$$

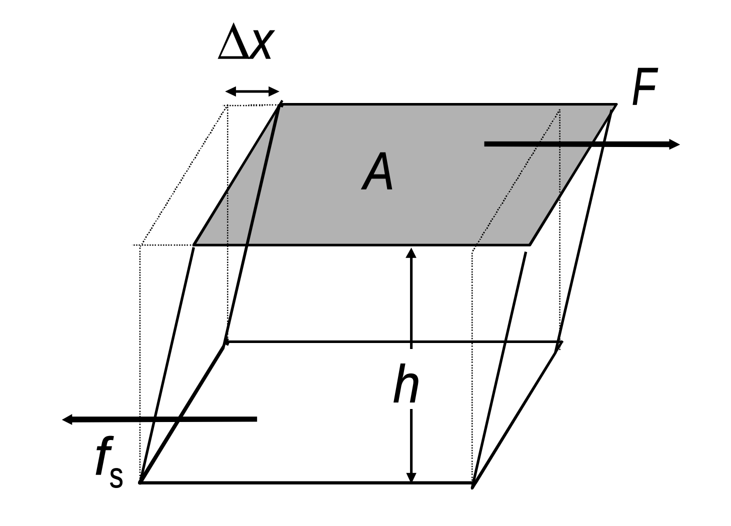


Experiments show that:

The figure shows that when the stress exceeds the elastic limit, the object is permanently distorted and will not return to its original shape after the stress is removed. As the stress is increased even further, the material will ultimately break.

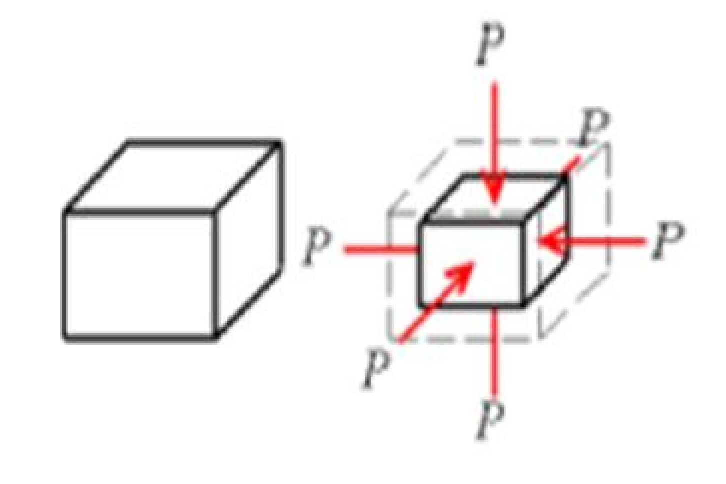


Shear Modulus: Shear's modulus measures the resistance of a solid to a change in its shape, i.e.,



$$S = \frac{shear\ stress}{shear\ strain} = \frac{F/A}{\Delta x/h}$$

Bulk Modulus: Bulk's modulus measures the resistance of a solid to a change in its volume, i.e.,



$$B = \frac{bulk\ stress}{bulk\ strain} = -\frac{F/A}{\Delta V/V} = -\frac{\Delta P}{\Delta V/V}$$

With ΔP is the change in pressure. The negative sign in the last equation is to ensure that *B* is a positive number. Because an increase in pressure causes a decrease in volume (negative Δ *V)* and vice versa.

Example 11.5 A mass of 100 kg is supported by a wire of length 2 m and cross-sectional area 0.1 cm2 . The wire is stretched by 0.22 cm. Find the tensile stress, the tensile strain, and the Young's modulus for the wire from this information.

Solution

$$Tensilestress = \frac{F}{A} = \frac{Mg}{A} = \frac{100 \times 9.8}{0.1 \times 10^{-4}} = 9.8 \times 10^{7}\ \mathrm{N/m^{2}}$$

$$Tensilestrain = \frac{\Delta L}{L} = \frac{0.22 \times 10^{-2}}{2} = 0.11 \times 10^{-2}$$

$$\Rightarrow\ Y = \frac{tensile\ stress}{tensile\ strain} = \frac{9.8 \times 10^{7}}{0.11 \times 10^{-2}} = 8.9 \times 10^{10}\ \mathrm{N/m^{2}}$$

Example 11.6 A solid lead sphere of volume 0.5 m3 is lowered to a depth in the ocean where the water pressure is equal to . The bulk modulus of lead is equal to . What is the change in volume of the sphere?

Solution

$$B = -\frac{\Delta P}{\Delta V / V} \quad \Rightarrow$$

$$\Delta V = -\frac{(\Delta P)V}{B}$$

$$= -\frac{0.5 \times 2 \times 10^{7}}{7.7 \times 10^{9}} = -1.3 \times 10^{-3}\ \mathrm{m}^{3}$$

The negative sign indicates a decrease in volume.

CHAPTER 12
OSCILLATORY MOTION

Before starting the discussion of the chapter's concepts it is worth to define some terms we will use frequently in this chapter:

1. The **period** of the motion, T, is the time required to complete one revolution.

2. The **frequency**, f, is the number of revolutions per unit time. The frequency is the reciprocal of the period, i.e., $f=T^{-1}$. The SI unit of frequency is hertz (Hz) with 1 Hz = 1 s^{-1}.

3. The **amplitude**, A, is the maximum displacement from some equilibrium position.

12.1 SIMPLE HARMONIC MOTION (SHM)

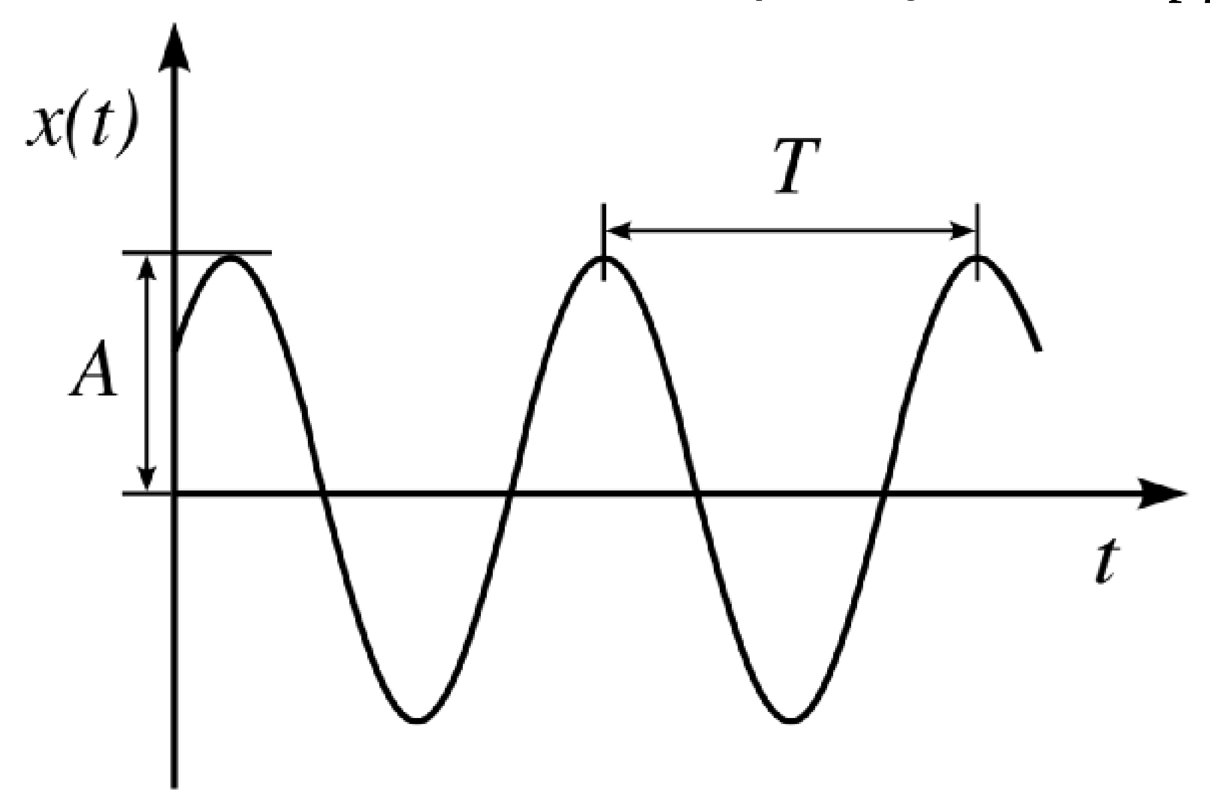
The simple harmonic motion is the motion that repeats itself in a particular way.

This means that for SHM motion the displacement x of the particle from equilibrium varies with time according to the relation

$$x = A\cos(wt + d)$$

where *A*, ω, and δ, are constants defined as:

A is the amplitude of the motion. ω is the angular frequency of the motion, and δ is the phase constant of the motion. The angular frequency ω is related to the linear frequency as $w=2pf$



The velocity of SHM:

differentiating the last equation with respect to $t \Rightarrow$:

$$v = \frac{dx}{dt} = -wA\sin(wt + d)$$

from the last equation we get: $v_{\max} = \pm wA$

The acceleration of SHM:

differentiating the last equation again with respect to $t \Rightarrow$:

$$a = \frac{dv}{dt} = -w^2 A\cos(wt + d) \Rightarrow$$

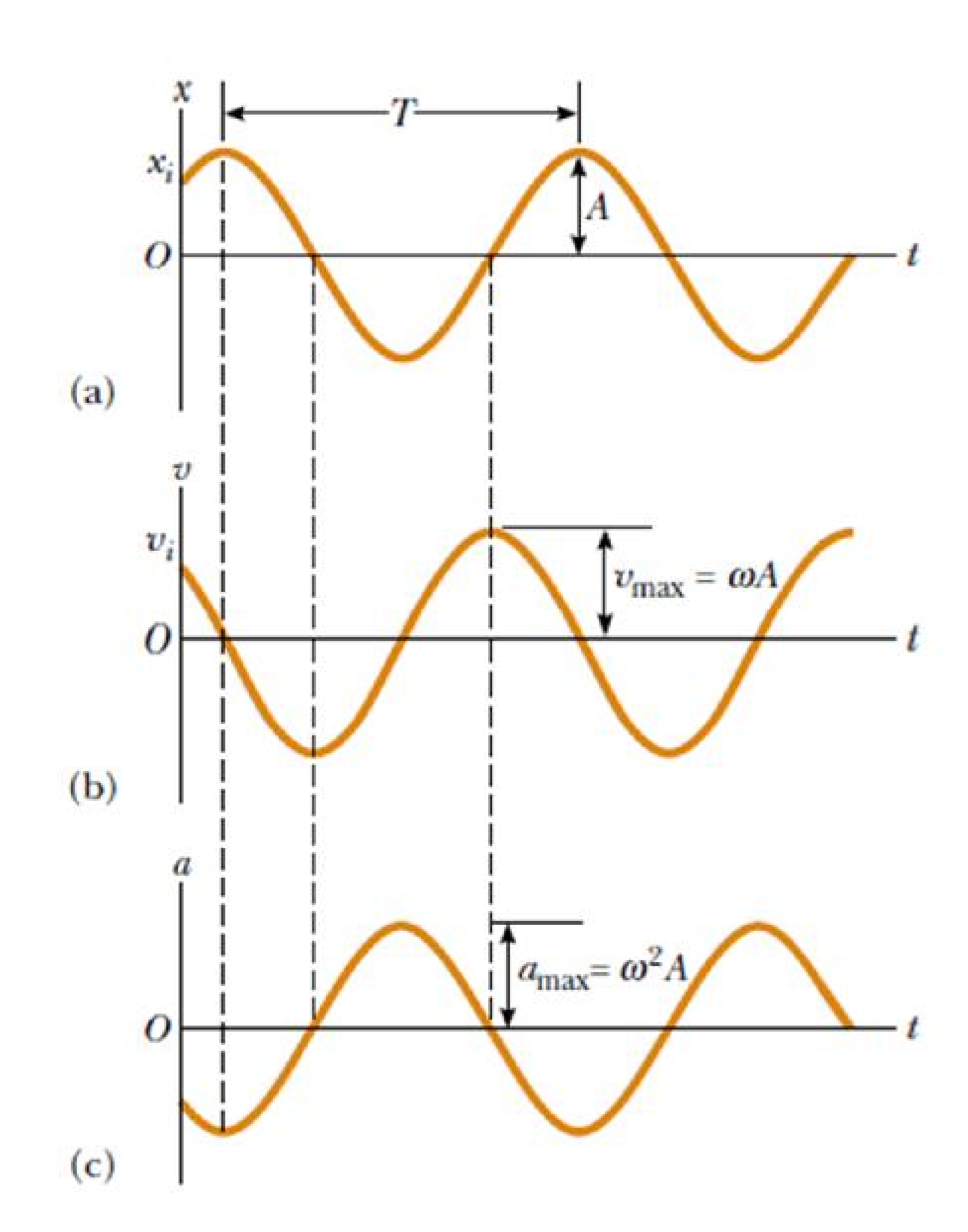
x

$$a = -w^2 x$$

Displacement, velocity and acceleration are plotted versus time. From these curves we note that:

(1) There is a phase difference of $\pi/2$ between *x* & *v*. That is, when *x* is a maximum or a minimum, the velocity is zero. Likewise when *x* is zero *v* is maximum or minimum.

(2) The phase of the acceleration differs from the phase of the displacement by π rad. That is, when *x* is a maximum, *a* is a maximum in the opposite direction.



From Newton's second law we have

$F = ma$ but $a = -w^2 x$ $\Rightarrow$ $F = -mw^2 x = -kx$ $\Rightarrow$

with $k = mw^2$ is constant

Since $a = d^2x \big/ dt^2$ $\Rightarrow$ $d^2x \big/ dt^2 = -w^2 x$ $\Rightarrow$

$$\frac{d^2x}{dt^2} + w^2 x = 0$$

Which is the differential equation of the simple harmonic motion.

It is easy to show that the solution of the last equation is.

$$x = A\cos(wt + d)$$

Now the SHM can be redefined as the motion broduced by a force given by

$$F = -kx = -mw^2 x$$

Example 12.1 The displacement of a particle varies with time according to

$$x=(6.0\text{m})\cos\left(pt+\frac{p}{3}\right)$$

where *t* in s, and the angles in the parentheses are in radians.

a) What is the amplitude and the frequency of the motion?

b) Determine the position, velocity, and acceleration of the particle at *t* = 2 s.

c) Determine the maximum speed and maximum acceleration of the particle.

Solution a) comparing with the eq. $x = A\cos(wt+d) \Rightarrow$

$$A = 6.0\,m,\ \ w = 3.14\,\text{rad/s, and} \qquad d = \frac{p}{3}$$

b) Substituting for *t*=2 s $\Rightarrow$

$$x_{t=2\text{s}}=(6.0\text{m})\cos\left(2p+\frac{p}{3}\right)=3.0\,\text{m}$$

$$v = \frac{dx}{dt} = -(6.0p)\sin\left(pt + \frac{p}{3}\right) \quad \Rightarrow$$

$$v_{t=2\,\mathrm{s}} = -(6.0p)\sin\left(2p + \frac{p}{3}\right) = -16.3\,\mathrm{m/s}$$

$$a = \frac{dv}{dt} = -(6.0p^2)\cos\left(pt + \frac{p}{3}\right) \quad \Rightarrow$$

$$a_{t=2\,\mathrm{s}} = -(6.0p^2)\cos\left(2p + \frac{p}{3}\right) = -29.6\,\mathrm{m/s}^2$$

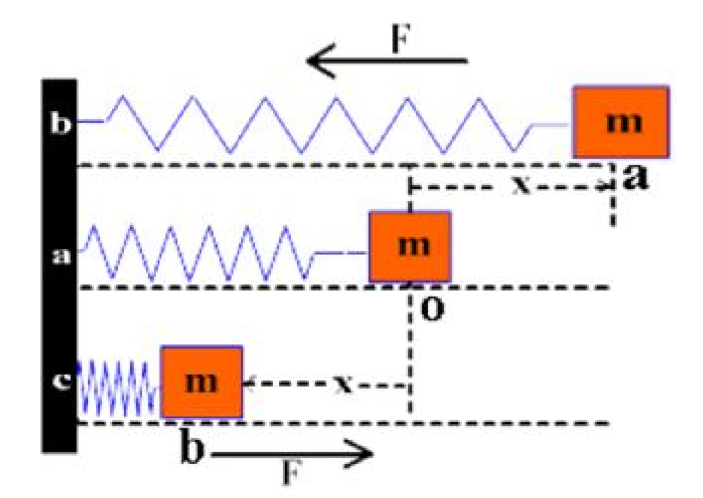
c)

$$v_{\max} = Aw = 6.0p = 10.8\,\mathrm{m/s}$$

$$a_{\max} = Aw^2 = 6.0p^2 = 59.2\,\mathrm{m/s}^2$$

12.2 MASS ATTACHED TO A SPRING

As a first example of the simple harmonic motion, we consider the system of a mass attached to a spring.



Since the spring exerts a force *F* given by

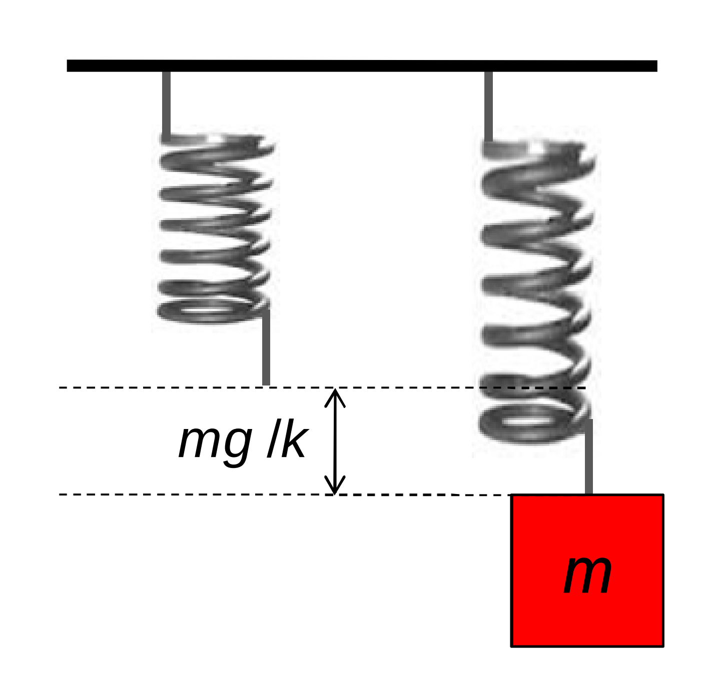
$$F = -kx$$

This means that our system will exhibit a simple harmonic motion with frequency and period given by

$$w = \sqrt{\frac{k}{m}} \qquad T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}}$$

It is interesting to note that a mass suspended from a vertical spring attached to a fixed point will also exhibit simple harmonic motion. The total force now acting on the mass is

$$F = -kx + mg = -k\left(x - \frac{mg}{k}\right)$$



Measuring the displacement relative to the new equilibrium position by letting

$$(x - mg/k) \to x \qquad \Rightarrow$$

$$F = -kx'$$

Example 12.2 Consider a mass-spring system with mass m=0.2 kg and spring constant k=5 N/m. The mass is displaced a distance 0.05 m, and then released from rest.

a) Find T and ω.

b) Find the maximum speed and maximum acceleration

c) Write down the displacement, the speed, and the acceleration as function of time.

Solution a) The angular frequency is

$$w = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{0.2}} = 5\,\text{rad/s}$$

And the period is $$T = \frac{2p}{w} = \frac{2p}{5} = 1.26\,s$$

b) It is clear that the amplitude *A*=0.05 m, then

$$v_{\text{max}} = wA = (5)0.05 = 0.25\,\text{s} \qquad a_{\text{max}} = w^2 A = (5)^2 \times 0.05 = 1.25\,\text{m/s}^2$$

b) The displacement is given by $x = A\cos(wt + d)$

But at t=0, x=A=0.05 m, $\Rightarrow A = A\cos(d) \quad \Rightarrow \cos(d) = 1 \quad \Rightarrow d = 0$

$$x = A\cos(wt) = \left(0.05\,\text{m}\right)\cos\left(5t\right)$$

$$v = \frac{dx}{dt} = -(0.25)\sin\left(5t\right)$$

$$a = \frac{dv}{dt} = -(1.25)\cos\left(5t\right)$$

12.3 ENERGY OF SHM

The potential energy is given by $U = \frac{1}{2}kx^2$

For SHM with d=0 we have $x = A\cos(wt) \Rightarrow U = \frac{1}{2}kA^2\cos^2(wt)$

The kinetic energy is given by $K = \frac{1}{2}mv^2$

For SHM with d=0 we have $v = -wA\sin(wt) \Rightarrow$

$$K = \frac{1}{2}\left(mw^2\right)A^2\sin^2(wt) = \frac{1}{2}kA^2\sin^2(wt)$$

The mechanical energy is now

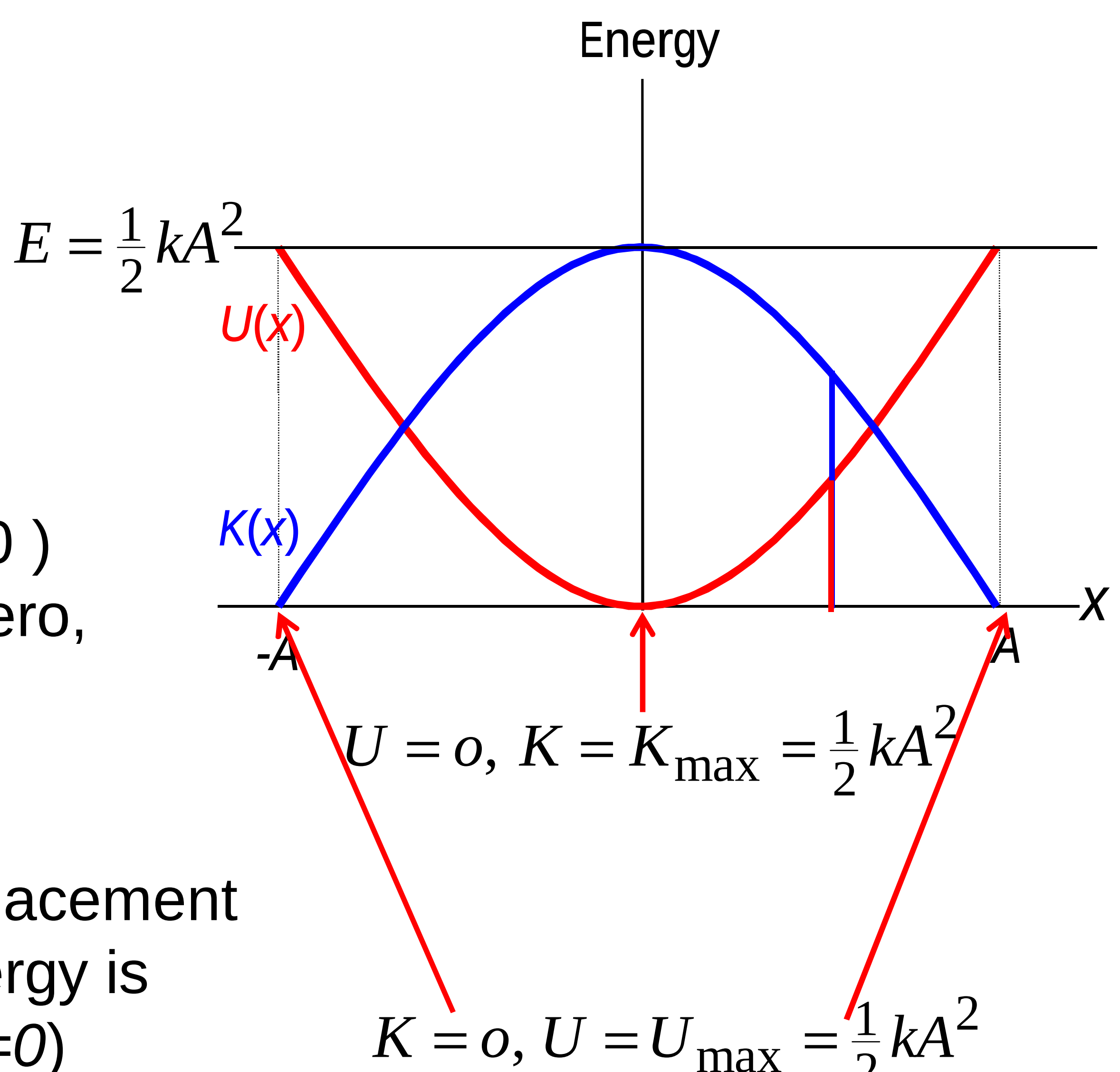
$$E = K + U = \frac{1}{2}kA^2\left[\sin^2(wt) + \cos^2(wt)\right] = \frac{1}{2}kA^2$$

This means that the mechanical energy is constant (conserved).

Plotting U and K versus x we obtain

The sum of U and K at any point is always constant and equal to

$$E = \frac{1}{2} k A^2$$



(i) At the equilibrium (x=0) the potential energy is zero, but the kinetic energy is maximum (K=E)

(ii) At the maximum displacement (x=±A) the potential energy is maximum (U=E), but (K=0)

To obtain the velocity of the particle at arbitrary displacement x using the conservation of energy we write

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \quad \Rightarrow$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \qquad = \pm w\sqrt{(A^2 - x^2)}$$

It is clear from the last equation that v=0 at x=±A, and v= ±Aw (maximum).

Example 12.3 A 200 g mass is attached to a spring and executes simple harmonic motion with a period of 0.25 s. If the total energy is 2 J. Find; **a)** The force constant of the spring
b) The amplitude of the motion.

Solution a) Using the relation $T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}} \Rightarrow$

$$k = \frac{(2p)^2 m}{T^2} = \frac{(2p)^2 \times 0.2}{(0.25)^2} = 126.3\,\text{N/m}$$

b) Now using the relation $E = \frac{1}{2}kA^2 \quad \Rightarrow \quad A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2\times 2}{126.3}} = 17.8\,\text{cm}$

Example 12.4 A mass of 0.2 kg is connected to a light spring of force constant 20 N/m oscillates horizontally on a smooth surface.

a) If the amplitude is *A*=2 cm, find the total energy of the system

b) Find the maximum speed.

c) What is the velocity when the displacement is equal to 1 cm.

d) Find the potential energy and the kinetic energy when the displacement equals 1 cm.

Solution a) The total energy is

$$E=\frac{1}{2}kA^2 \quad =\frac{1}{2}(20)(2\times10^{-2})^2 = 40\times10^{-4}\ \text{J}$$

b) It is known that $\frac{1}{2}mv_{max}^2 = E \Rightarrow \quad v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{80\times10^{-4}}{0.2}} = 0.2\,\text{m/s}$

c) Using $v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \Rightarrow = \sqrt{\frac{20}{0.2}((0.02)^2 - (0.01)^2)} = 0.17\,\text{m}$

d) We have $\quad U = \frac{1}{2}kx^2 = \frac{1}{2}(20)\left(1\times10^{-2}\right)^2 = 1\times10^{-3}\ \text{J}$

$$\text{Using } E = K + U \Rightarrow \quad K = E - U = 4\times10^{-3} - 1\times10^{-3} = 3\times10^{-3}\ \text{J}$$

12.4 ANGULAR SHM

It was shown that if a particle is under a force of the form *F=-kx*, this particle exhibits SHM.

The rotational analog of this equation *is* t=-*k*\`$q$. The constant *k*\` is called the torque constant.

$$\text{But } t = I a = I\frac{d^2 q}{dt^2} \Rightarrow \quad I\frac{d^2 q}{dt^2} = -k' q \Rightarrow \frac{d^2 q}{dt^2} + \frac{k'}{I} q = 0 \Rightarrow$$

$$\frac{d^2 q}{dt^2} + w^2 q = 0$$

Which is the rotational analog of the equation $\frac{d^2 x}{dt^2} + w^2 x = 0$

$$\text{with } w = \sqrt{\frac{k'}{I}} \quad \text{and } T = 2p\sqrt{\frac{I}{k'}}$$

12.5 THE SIMPLE PENDULUM

The simple pendulum consists of a point mass *m* suspended by a light string of length *L* as shown.

The tangential component *mg*sinθ is the restoring force, because it is always acts opposite to the displacement *s* so as to return the mass to the equilibrium position

Hence, the restoring force is

$$F = -mg\sin\theta$$

But $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$ $\Rightarrow$ for small θ $\quad \sin\theta \approx \theta \quad \Rightarrow$

$F = -mg\theta = -\frac{mg}{L}S$ $\quad \Rightarrow \quad$ We have SHM with $\quad k = \frac{mg}{L} \quad \Rightarrow$

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{mg/L}{m}}=\sqrt{\frac{g}{L}}$$

and the period is $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{L}{g}}$

Notice that the period and the frequency of a simple pendulum depend only on *L* and *g* and doesn't depend on *m*.

The torque acting on the mass about the pint of suspension is:

$$\tau=-mgL\sin\theta$$

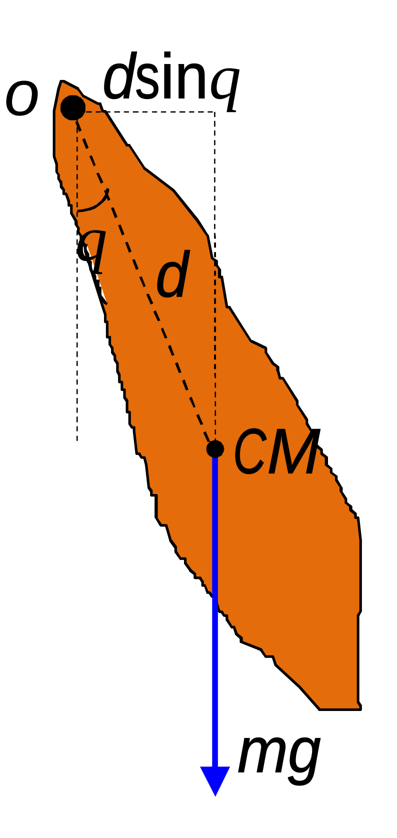
But for small θ $\quad \sin\theta\approx\theta \quad \Rightarrow \quad \tau=-(mgL)\theta$

Comparing with the relation $\quad \tau=k'\theta \;\Rightarrow\; k'=mgL \;\Rightarrow$

$$\omega=\sqrt{\frac{mgL}{I}}=\sqrt{\frac{mgL}{(mL^2)}} \Rightarrow \omega=\sqrt{\frac{g}{L}}$$

12.5 THE COMPOUND (PHYSICAL) PENDULUM

Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a physical pendulum or compound pendulum.



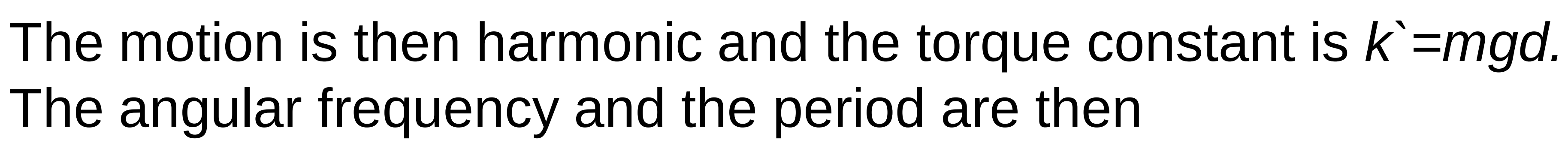
The torque about *O* is provided by the force of gravity as

$$\tau = -mgd\sin\theta$$

If θ is small, we can again approximate $\sin\theta$ by θ. With this approximation the last equation reads

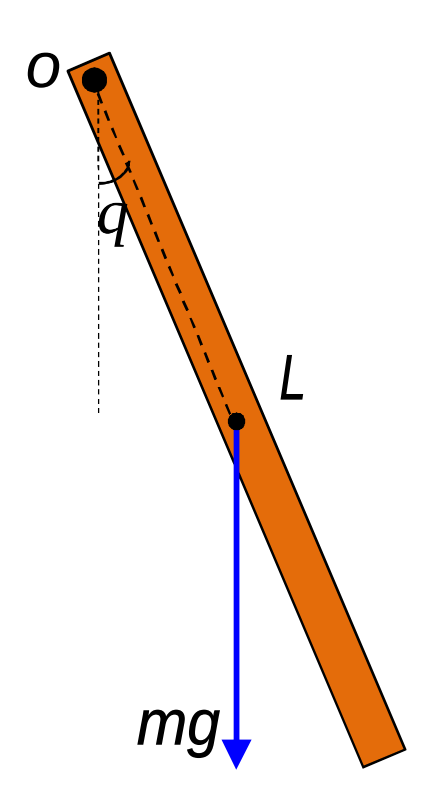
$$\tau = -(mgd)\theta$$

The motion is then harmonic and the torque constant is $k' = mgd$. The angular frequency and the period are then



$$\omega = \sqrt{\frac{k'}{I}} = \sqrt{\frac{mgd}{I}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Example 12.4 A uniform rod of length 1 m is oscillates in a vertical plane about one end with simple harmonic motion as shown. Calculate the angular frequency and the period of the motion.



Solution The moment of inertia for a rod about one end is

$$I = \tfrac{1}{3} ML^2$$

Since the rod is uniform, then the distance from the center of mass to the pivot is L/2 ⇒

$$w = \sqrt{\frac{mgd}{I}} = \sqrt{\frac{Mg(L/2)}{\frac{1}{3}ML^2}} = \sqrt{\frac{3g}{2L}} = \sqrt{\frac{3\times 9.8}{2\times 1}} = 3.83\,\text{rad/s}$$

$$T = \frac{2p}{w} = 2p\sqrt{\frac{2L}{3g}} = 1.64\,\text{s}$$

12.6 THE DAMPED OSCILLATIONS

Let us assume that the frictional force is proportional to the velocity, which is the case for most of the retarding forces, that is

$$F = -bv$$

where *v* is the velocity and *b* is a constant, the total force acting on the body is then

$$\sum F = -kx - bv \quad \Rightarrow \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

If *b* is relatively small, the solution of the above equation is

$$x = \left(Ae^{-(b/2m)t}\right)\cos(wt + d)$$

$$\text{with } w = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

